

FOURIER ANALYSIS & METHODS

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ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications*, by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2019.01.31

1.1. Example of the vibrating string. Assume that at $t = 0$, the ends of the string are fixed, and we have pulled up the middle of it. This makes a shape which mathematically is described by the function

$$v(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ 2\pi - x, & \pi \leq x \leq 2\pi \end{cases}$$

Assume that at $t = 0$ the string is not yet vibrating, so the initial conditions are then

$$\begin{cases} u(x, 0) = v(x) \\ u_t(x, 0) = 0 \end{cases}$$

We assume the ends of the string are fixed, so we have the boundary conditions

$$u(0) = u(2\pi) = 0.$$

The string is identified with the interval $[0, 2\pi]$. Determine the function $u(x, t)$ which gives the height at the point x on the string at the time $t \geq 0$ which satisfies all these conditions.

1.1.1. First Step: Separate Variables. We use our first technique, separation of variables. The wave equation demands that

$$\square u = 0, \quad \square u = \partial_{tt}u - \partial_{xx}u.$$

Write

$$u(x, t) = X(x)T(t).$$

Hit it with the wave equation:

$$X(x)T''(t) - X''(x)T(t) = 0.$$

We again *separate the variables* by dividing the whole equation by $X(x)T(t)$. Then we have

$$\frac{T''(t)}{T(t)} - \frac{X''(x)}{X(x)} = 0 \implies \frac{T''}{T} = \frac{X''}{X} = \text{constant}.$$

The two sides depend on different variables, which makes them both have to be constant. We give that a name, λ . Then, since we have those handy dandy boundary

conditions for X (but a much more complicated initial condition for $u(x, 0) = v(x)$) we start with X . We solve

$$X'' = \lambda X, \quad X(0) = X(2\pi) = 0.$$

Exercise 1. Show that the cases $\lambda \geq 0$ won't satisfy the boundary condition.

We are left with $\lambda < 0$ which by our multivariable calculus theorem tells us that

$$X(x) = a \cos(\sqrt{|\lambda|x}) + b \sin(\sqrt{|\lambda|x}).$$

To get $X(0) = 0$, we must have $a = 0$. To get $X(2\pi) = 0$ we will need

$$\sqrt{|\lambda|}2\pi = k\pi \quad k \in \mathbb{Z}.$$

Hence

$$\sqrt{|\lambda|} = \frac{k}{2}, \quad k \in \mathbb{Z}.$$

Since $\sin(-x) = -\sin(x)$ are linearly dependent, we only need to take $k \in \mathbb{N}$ (without 0, you know, American \mathbb{N}). So, we have X which we index by n , writing

$$X_n(x) = \sin(nx/2) \quad n \in \mathbb{N}.$$

For now, we don't worry about the constant factor. Next, we have the equation for the partner-function (can't forget the partner function!)

$$\frac{T_n''}{T_n} = \lambda_n.$$

Since we know that $\lambda_n < 0$ and $\sqrt{|\lambda_n|} = n/2$ we have

$$\lambda_n = -\frac{n^2}{4}.$$

Hence, our handy dandy multivariable calculus theorem tells us that the solution

$$T_n(t) = a_n \cos(nt/2) + b_n \sin(nt/2).$$

Now, we have

$$u_n(x, t) = X_n(x)T_n(t), \quad \square u_n = 0 \quad \forall n \in \mathbb{N}.$$

1.1.2. *Supersolution obtained by superposition principle.* Since the PDE is linear and homogeneous, we also have

$$\square \sum_{n \geq 1} u_n(x, t) = \sum_{n \geq 1} \square u_n(x, t) = 0.$$

We don't know which of these u_n we need to build our solution according to the initial conditions, so we just take all of them for now.

1.1.3. *Fourier series to find the coefficients using the initial conditions.* We need

$$u(x, t) := \sum_{n \geq 1} u_n(x, t)$$

to satisfy the initial conditions. The first is that

$$u(x, 0) = \sum_{n \geq 1} X_n(x)a_n = v(x).$$

We are working on the interval $[0, 2\pi]$. The coefficients are obtained by using X_n as a basis for \mathcal{L}^2 on this interval. The coefficients are therefore

$$\boxed{\text{an1}} \quad (1.1) \quad a_n = \frac{1}{\|X_n\|^2} \langle v, X_n \rangle = \frac{\int_0^{2\pi} v(x) \overline{X_n(x)} dx}{\int_0^{2\pi} |X_n(x)|^2 dx}.$$

If one wishes to do these integrals, one is welcome to do so. That will not be necessary on the exam, however.

To obtain the b_n coefficients, we use the other initial condition which says that

$$\begin{aligned} u_t(x, 0) &= \sum_{n \geq 1} X_n(x) T'_n(0) = \sum_{n \geq 1} X_n(x) \left(-a_n \frac{n}{2} \sin(0) + b_n \frac{n}{2} \cos(0) \right) \\ &= \sum_{n \geq 1} X_n(x) \frac{n}{2} b_n = 0. \end{aligned}$$

These coefficients are calculated in the same way:

$$\frac{n}{2} b_n = \frac{\langle 0, X_n \rangle}{\|X_n\|^2} = 0 \forall n.$$

Hence, our solution is

$$\sum_{n \geq 1} a_n \sin(nx/2) \cos(nt/2),$$

with a_n given in equation $\boxed{\text{an1}}$ (I.I).

1.2. Summary of methods for solving PDEs on bounded intervals. Thus far we have collected the following techniques to solve PDEs like the heat and wave equation on *bounded* intervals:

- (1) Separation of variables (a means to an end),
- (2) Superposition position (smash solutions together to make a supersolution),
- (3) Fourier series to find the coefficients obtained using the initial data (\mathcal{L}^2 scalar product and divide by the norm).

These methods work well on *bounded* intervals.

1.3. Another wave equation example. Solve:

$$\begin{aligned} u_{tt} &= u_{xx}, \quad t > 0, \quad x \in (-1, 1), \\ \left\{ \begin{array}{l} u(0, x) &= 1 - |x| \\ u_t(0, x) &= 0 \\ u_x(t, -1) &= 0 \\ u_x(t, 1) &= 0 \end{array} \right. \end{aligned}$$

We use separation of variables, writing $u(x, t) = X(x)T(t)$. It is just a means to an end. We write the PDE:

$$T''X = X''T.$$

Divide everything by XT to get

$$\frac{T''}{T} = \frac{X''}{X}.$$

Since the two sides depend on different variables, they are both constant. Start with the X side because we have more simple information about it. The boundary conditions that

$$u_x(t, -1) = u_x(t, 1) = 0 \implies X'(-1) = X'(1) = 0.$$

So, we have the equation

$$\frac{X''}{X} = \text{constant, call it } \lambda.$$

Thus we are solving

$$X'' = \lambda X, \quad X'(-1) = X'(1) = 0.$$

Case 1: $\lambda = 0$: In this case, we have solved this equation before. One way to think about it is like the second derivative is like acceleration. If $X'' = 0$, it's like saying X has constant acceleration. Therefore X can only be a linear function. Now, we have the boundary condition which says that $X'(-1) = X'(1) = 0$. So the slope of the linear function must be zero, hence X must be a constant function in this case. So, the only solutions in this case are the constant functions.

Case 2: $\lambda > 0$: In this case, a general solution is of the form:

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}.$$

Let us assume that A and B are not both zero. The left boundary condition requires

$$A\sqrt{\lambda}e^{-\sqrt{\lambda}} - \sqrt{\lambda}Be^{\sqrt{\lambda}} = 0.$$

Since $\lambda > 0$ we can divide by $\sqrt{\lambda}$ to say that we must have

$$Ae^{-\sqrt{\lambda}} = Be^{\sqrt{\lambda}} \implies \frac{A}{B} = e^{2\sqrt{\lambda}}.$$

The right boundary condition requires

$$A\sqrt{\lambda}e^{\sqrt{\lambda}} - \sqrt{\lambda}Be^{-\sqrt{\lambda}} = 0.$$

Since $\lambda > 0$, we can divide by $\sqrt{\lambda}$, to make this:

$$Ae^{\sqrt{\lambda}} = Be^{-\sqrt{\lambda}} \implies e^{2\sqrt{\lambda}} = \frac{B}{A}.$$

Hence combining with the other boundary condition we get:

$$\frac{A}{B} = e^{2\sqrt{\lambda}} = \frac{B}{A} \implies A^2 = B^2 \implies A = \pm B \implies \frac{A}{B} = \pm 1.$$

Neither of these are possible because

$$e^{2\sqrt{\lambda}} > 1 \text{ since } 2\sqrt{\lambda} > 0.$$

So, we run amok under the assumption that A and B are not both zero. Hence, the only solution in this case requires $A = B = 0$. This is the waveless wave.

Case 3: $\lambda < 0$: In this case a general solution is of the form:

$$X(x) = a \cos(\sqrt{|\lambda|x}) + b \sin(\sqrt{|\lambda|x}).$$

To satisfy the left boundary condition we need

$$-a\sqrt{|\lambda|} \sin(-\sqrt{|\lambda|}) + b\sqrt{|\lambda|} \cos(-\sqrt{|\lambda|}) = 0 \iff a \sin(\sqrt{|\lambda|}) = -b \cos(\sqrt{|\lambda|}).$$

To satisfy the right boundary condition we need

$$-a\sqrt{|\lambda|} \sin(\sqrt{|\lambda|}) + b\sqrt{|\lambda|} \cos(\sqrt{|\lambda|}) = 0 \iff a \sin(\sqrt{|\lambda|}) = b \cos(\sqrt{|\lambda|}).$$

Hence we need

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$$(1.2) \quad a \sin(\sqrt{|\lambda|}) = -b \cos(\sqrt{|\lambda|}) = b \cos(\sqrt{|\lambda|}).$$

We do not want both a and b to vanish. So, we need to have either

(1) the sine vanishes, so we need $\sin(\sqrt{|\lambda|}) = 0$ which then implies that

$$\sqrt{|\lambda|} = n\pi, \quad n \in \mathbb{Z}$$

(2) or the cosine vanishes so we need $\cos(\sqrt{|\lambda|}) = 0$ which then implies that

$$\sqrt{|\lambda|} = \left(n + \frac{1}{2}\right)\pi, \quad n \in \mathbb{N}.$$

Note that these two cases are *mutually exclusive*. In case (1), by (1.2) this means that $b = 0$. In case (2), by (1.2) this means that $a = 0$. So, we have two types of solutions, which up to constant factor look like:

$$X_m(x) = \begin{cases} \cos(m\pi x/2) & m \text{ is even} \\ \sin(m\pi x/2) & m \text{ is odd} \end{cases}$$

In both cases,

$$\lambda_m = -\frac{m^2\pi^2}{4}.$$

We can now solve for the partner function, $T_m(t)$. The equation is

$$\frac{T_m''}{T_m} = \frac{X_m''}{X_m} = \lambda_m = -\frac{m^2\pi^2}{4}.$$

Therefore, we are in case 3 for the T_m function as well, so we know that

$$T_m(t) = a_m \cos\left(\frac{m\pi t}{2}\right) + b_m \sin\left(\frac{m\pi t}{2}\right).$$

Then we have for

$$u_m(x, t) = X_m(x)T_m(t), \quad \square u_m = 0 \quad \forall m.$$

(Recall that $\square = \partial_{tt} - \partial_{xx}$, that is the wave operator). Hence, our functions solve a homogeneous PDE, so we can use the superposition principle to smash them all together to make a super solution:

$$u(x, t) = \sum_{m \in \mathbb{N}} u_m(x, t) = \sum_{n \in \mathbb{N}} X_m(x) \left(a_m \cos\left(\frac{m\pi t}{2}\right) + b_m \sin\left(\frac{m\pi t}{2}\right) \right).$$

How do we determine the coefficients? Using the initial data and a Fourier series for it!!!

The initial data is

$$\begin{cases} u(0, x) &= 1 - |x| \\ u_t(0, x) &= 0 \end{cases}$$

Let us plug $t = 0$ into our solution:

$$u(x, 0) = \sum_{m \in \mathbb{N}} X_m(x) a_m.$$

We demand that this is the initial data, so we need

$$1 - |x| = \sum_{m \in \mathbb{N}} X_m(x) a_m.$$

It is a Fourier series on the right side!! We therefore just need to expand the function $1 - |x|$ in a Fourier series. If we think about the basis functions $\{X_m(x)\}_{m \geq 0}$ then

$$a_m = \frac{\langle 1 - |x|, X_m(x) \rangle}{\|X_m\|^2},$$

where

$$\langle 1 - |x|, X_m(x) \rangle = \int_{-1}^1 (1 - |x|) \overline{X_m(x)} dx,$$

$$\|X_m\|^2 = \int_{-1}^1 |X_m(x)|^2 dx.$$

On an exam, you are not actually required to compute these integrals!

Now, for the other coefficients (the b_n), we use the condition on the derivative:

$$u_t(x, 0) = \sum_{m \in \mathbb{N}} m_n \frac{m\pi}{2} X_m(x) = 0.$$

We know how to Fourier expand the zero function: its coefficients are all just zero. Hence, it suffices to take

$$b_m = 0 \forall m.$$

1.4. Fourier series on an arbitrary interval. When we use our tools to solve a PDE on a finite interval, as above, the initial data is *not* a periodic function. Moreover, it was not defined on the interval $(-\pi, \pi)$. The technique still works! It is actually quite beautiful. When we determined the coefficients, we solved for the Fourier coefficients on the interval $(-1, 1)$. Here we explain how to do that in general.

For a function f defined on an interval $[a - \ell, a + \ell]$ for some $a \in \mathbb{R}$, and some $\ell > 0$, we begin by extending f to be 2ℓ periodic on \mathbb{R} . Next, we define

$$g(t) := f\left(\frac{t\ell}{\pi} + a\right) = f(x),$$

that is

$$\frac{t\ell}{\pi} + a = x, \quad t = \frac{(x - a)\pi}{\ell}.$$

Then, the function $g(t)$ is 2π periodic, because

$$g(t + 2\pi) = f\left(\frac{(t + 2\pi)\ell}{\pi} + a\right) = f\left(\frac{t\ell}{\pi} + a + 2\ell\right) = f\left(\frac{t\ell}{\pi} + a\right).$$

Above, we used the fact that f is 2ℓ periodic. If g is in \mathcal{L}^2 , then we can expand it into a Fourier series:

$$\sum_{n \in \mathbb{Z}} c_n e^{int},$$

with coefficients

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-int} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{t\ell}{\pi} + a\right) e^{-int} dt.$$

Substituting in the integral,

$$x = \frac{t\ell}{\pi} + a, \quad dx = \frac{\ell dt}{\pi}$$

the coefficients become:

$$c_n = \frac{1}{2\pi} \frac{\pi}{\ell} \int_{a-\ell}^{a+\ell} f(x) e^{-in(x-a)\pi/\ell} dx = \frac{1}{2\ell} \int_{a-\ell}^{a+\ell} f(x) e^{-in(x-a)\pi/\ell} dx.$$

Then, we get by substituting for t in terms of x the Fourier series for f ,

$$\sum_{n \in \mathbb{Z}} c_n e^{in\left(\frac{(x-a)\pi}{\ell}\right)}.$$

The same relationship holds for the Fourier cosine and sine coefficients:

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}), \quad n \geq 1,$$

or equivalently

$$a_n = \frac{1}{\ell} \int_{a-\ell}^{a+\ell} f(x) \cos(n(x-a)\pi/\ell) dx, \quad b_n = \frac{1}{\ell} \int_{a-\ell}^{a+\ell} f(x) \sin(n(x-a)\pi/\ell) dx,$$

and the Fourier series has the form

$$\frac{a_0}{2} + \sum_{n \geq 1} a_n \cos(n(x-a)\pi/\ell) + b_n \sin(n(x-a)\pi/\ell).$$

To what does the Fourier series converge?

Theorem 1. Assume that f is defined on an interval $[a-\ell, a+\ell]$ for some $a \in \mathbb{R}$, and some $\ell > 0$, such that f is piecewise \mathcal{C}^1 on this interval. Then the Fourier series for f , defined by

$$\sum_{n \in \mathbb{Z}} c_n e^{in\left(\frac{x-a}{\ell}\pi\right)}, \quad c_n = \frac{1}{2\ell} \int_{a-\ell}^{a+\ell} f(x) e^{-in(x-a)\pi/\ell} dx,$$

or equivalently the series

$$\frac{a_0}{2} + \sum_{n \geq 1} a_n \cos(n(x-a)\pi/\ell) + b_n \sin(n(x-a)\pi/\ell)$$

converges to $f(x)$ for all $x \in (a-\ell, a+\ell)$ at which f is continuous. At a point $x \in (a-\ell, a+\ell)$ where f is not continuous, the series converges to

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$$(1.3) \quad \frac{f(x_+) + f(x_-)}{2}.$$

Exercise 2. Prove the theorem. As a hint: apply the Theorem PCF Σ to the function g above.

1.5. Two primary applications of Fourier series. We now have to main uses for Fourier series.

- (1) Solving PDEs on bounded intervals. This proceeds in three steps: (1) separation of variables (a means to an end), (2) smashing all solutions obtained in this way together to create a *super solution* (superposition), and (3) using a Fourier series to express the initial data.
- (2) Using Theorem 2.1 to compute nifty sums like:

$$\sum_{n \geq 1} \frac{1}{n^2}.$$

To compute such a sum, you will first compute the Fourier series of a certain function f which is defined on $(-\pi, \pi)$ and extended 2π periodically:

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

Next, substituting a specific value of x you want to recover the desired sum, like $\sum n^{-2}$. You use the theorem to conclude that the series converges to the average of the left and right limit of the function at x . Then re-arrange to obtain your desired sum.

The simplest way to compute the sum

$$\sum_{n \geq 1} \frac{1}{n^4}$$

requires deep theorems about Hilbert spaces, which is our next topic. These theorems will tell us that

$$\sum_{n \in \mathbb{Z}} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx,$$

for

$$f(x) := x^2 \text{ for } |x| \leq \pi, \text{ and extended to be } 2\pi \text{ periodic on } \mathbb{R},$$

with

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

If we have looked up the Fourier series (or we compute it), we find:

$$\frac{\pi^2}{3} + \sum_{n \geq 1} \frac{4(-1)^n \cos(nx)}{n^2}.$$

This is not given in terms of c_n but we can nonetheless obtain the c_n since:

$$a_n = c_n + c_{-n} = \frac{4(-1)^n}{n^2}, \quad b_n = i(c_n - c_{-n}) = 0 \forall n \geq 1 \implies c_n = c_{-n}$$

and thus

$$a_n = 2c_n \implies c_n = \frac{2(-1)^n}{n^2} = c_{-n} \quad \forall n \geq 1.$$

The magical Hilbert space theory therefore tells us that

$$\sum_{n \in \mathbb{Z}} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x^2|^2 dx = \frac{1}{2\pi} \frac{2\pi^5}{5} = \frac{\pi^4}{5}.$$

On the left side,

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |c_n|^2 &= |c_0|^2 + 2 \sum_{n \geq 1} |c_n|^2 = \frac{\pi^4}{9} + 2 \sum_{n \geq 1} \left| \frac{2(-1)^n}{2n^2} \right|^2 = \frac{\pi^4}{9} + 2 \sum_{n \geq 1} \frac{4}{4n^4} \\ &= \frac{\pi^4}{9} + 8 \sum_{n \geq 1} \frac{1}{n^4}. \end{aligned}$$

Consequently,

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8 \sum_{n \geq 1} \frac{1}{n^4} \implies \frac{\pi^4}{5} - \frac{\pi^4}{9} = 8 \sum_{n \geq 1} \frac{1}{n^4} \implies \frac{9\pi^4 - 5\pi^4}{8 * 45} = \sum_{n \geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Our main motivation for developing Hilbert space theory (in case we are not simply motivated by the love of the theory itself) are that this theory will:

- (1) provide new tools to be able to explicitly evaluate series using Fourier series (as done above);
- (2) determine if our solution found by the Fourier series method is indeed *the unique* solution to our PDE on a bounded interval;
- (3) provide new tools to be able to solve PDEs in other compact geometric settings (like in a rectangle, disk, annulus, cylinder, box, sphere, and so forth).

1.5.1. *Exercises to be done by oneself: Answers.*

- (1) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) := x(\pi - |x|).$$

Okay, it is

$$\sum_{n \geq 1} \frac{8 \sin((2n-1)x)}{\pi(2n-1)^3}.$$

- (2) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) = e^{bx}.$$

Okay, it is

$$\sum_{n \in \mathbb{Z}} \frac{\sinh(b\pi)(-1)^n}{\pi(b-in)} e^{inx}.$$

- (3) Use the Fourier series for the function $f(x) = |\sin(x)|$ to compute the sum

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi-2}{4}.$$

The Fourier series is

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n \geq 1} \frac{\cos(2nx)}{4n^2-1}.$$

So, to obtain the first sum, one can use $x = 0$. The series will converge to 0, so you get that

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n \geq 1} \frac{1}{4n^2-1} = 0.$$

Then, re-arranging, one obtains the desired sum. To get the sum with the $(-1)^{n+1}$ upstairs, one should use $x = \frac{\pi}{2}$, because then upstairs one has

$$\cos(2n\pi/2) = \cos(n\pi) = (-1)^n.$$

The series will converge to $|\sin(\pi/2)| = 1$. The same idea applies to re-arrange and obtain the desired sup.

- (4) Use the Fourier series for the function $f(x) = x(\pi - |x|)$ to compute the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

We have computed the Fourier series above. The question now is what value of x to use? Well, upstairs we have

$$\sin((2n-1)x).$$

For $x = \pi/2$ this becomes

$$\sin((2n-1)\pi/2).$$

This will alternate between +1 like when $n = 1$ and -1 like when $n = 2$. So, we can compute in this way that

$$\sin((2n-1)\pi/2) = (-1)^{n+1}.$$

Consequently, for $x = \pi/2$ the series is

$$\sum_{n \geq 1} \frac{8(-1)^{n+1}}{\pi(2n-1)^3}.$$

It converges to the average of the left and right limits of $f(x)$ at $x = \pi/2$. These are the same and are both equal to

$$\frac{\pi^2}{4}.$$

Hence

$$\frac{\pi^2}{4} = \sum_{n \geq 1} \frac{8(-1)^{n+1}}{\pi(2n-1)^3}.$$

Re-arrange to get the desired sum.

- (5) Let $f(x)$ be the periodic function such that $f(x) = e^x$ for $x \in (-\pi, \pi)$, and extended to be 2π periodic on the rest of \mathbb{R} . Let

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

be its Fourier series. Therefore, by Theorem 2.1

$$e^x = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad \forall x \in (-\pi, \pi).$$

If we differentiate this series term-wise then we get $\sum inc_n e^{inx}$. On the other hand, we know that $(e^x)' = e^x$. So, then we should have

$$\sum inc_n e^{inx} = \sum c_n e^{inx} \implies c_n = inc_n \quad \forall n.$$

This is clearly wrong. Where is the mistake?

DO NOT DIFFERENTIATE THE SERIES TERMWISE!!! That's the mistake. One can only differentiate termwise when the function satisfies the hypotheses of Theorem 2.3. That theorem requires the function to be continuous on \mathbb{R} . The function e^x on $(-\pi, \pi)$ and extended to be 2π periodic on \mathbb{R} has discontinuities at $\pi + 2n\pi$ for all $n \in \mathbb{Z}$. So it fails to satisfy the hypotheses of the theorem, thus that theorem does not apply to this function.

- (6) Determine the Fourier sine and cosine series of the function

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

Okay, they are

$$\frac{\pi}{4} - \frac{2}{\pi} \sum_{n \geq 1} \frac{\cos((4n-2)x)}{(2n-1)^2}, \quad \frac{4}{\pi} \sum_{n \geq 1} (-1)^{n+1} \frac{\sin((2n-1)x)}{(2n-1)^2}.$$

- (7) Expand the function

$$f(x) = \begin{cases} 1 & 0 < x < 2 \\ -1 & 2 < x < 4 \end{cases}$$

in a cosine series on $[0, 4]$. Okay, it is

$$\frac{4}{\pi} \sum_{n \geq 1} \frac{(-1)^{n+1}}{2n-1} \cos\left(\frac{(2n-1)\pi x}{4}\right).$$

(8) Expand the function e^x in a series of the form

$$\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x}, \quad x \in (0, 1).$$

Okay, it is

$$(e-1) \sum_{n \in \mathbb{Z}} \frac{e^{2\pi i n x}}{1-2\pi i n}.$$

(9) Define

$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 1 & 1 < t < 2 \\ 3-t & 2 \leq t \leq 3 \end{cases}$$

and extend f to be 3-periodic on \mathbb{R} . Expand f in a Fourier series. Determine, in the form of a Fourier series, a 3-periodic solution to the equation

$$y''(t) + 3y(t) = f(t).$$

This is Extra Exercise 2, and the solution is contained in the extra övningar document on the course homepage.