Recall the Fourier transform of a 2\pi-periodic function:
\[ \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt = \langle f, e^{i\omega \cdot} \rangle. \]

Defining \[ \langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} \, dt. \]

The Fourier transform gives a decomposition in par frequencies \( e^{i\omega t} \):
\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega. \]

The Wavelet decomposition is different.

Let \( \varphi(t) \) be a given function and let
\[ \varphi_{j,k} = \frac{1}{2^j} \varphi(2^j t - k) \]
\( \uparrow \) dilation translation.

Under suitable conditions
\[ f(t) = \sum_{j,k=-\infty}^{\infty} \langle f, \varphi_{j,k} \rangle \varphi_{j,k}(t). \]

Example: The Haar Wavelet.

Let \( \varphi(t) = \varphi(2t) + \varphi(2t-1) \left\{ \begin{array}{ll} \text{need to check that} & \text{it is possible} \\
\bar{\varphi}(t) = \varphi(2t) - \varphi(2t-1) \left\{ \begin{array}{ll} \text{can always be} & \text{achieved} \\
\end{array} \right. \\
\end{array} \right. \]
\( \varphi \) is called a scaling function, \( \psi \) a wavelet.

The Haar scaling function is:
\[ \varphi(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{else} \end{cases} \]

and the wavelet is:
\[ \psi(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{else} \end{cases} \]

Note that \( \langle \varphi, \psi \rangle = 0 \)
\[ \langle \varphi(t-k), \psi(t-h) \rangle = 0, (n+k) \]
\[ \langle \psi(t-k), \psi(t-h) \rangle = 0, (n+k) \]
Example

Let \( f(t) = \begin{cases} \frac{t^2}{2} & 0 < t < 1 \\ 0 & \text{else} \end{cases} \)

The function \( f(t) \) can be approximated by step function, which is the mean \( f \) in each interval.

For example in the interval \( [\frac{1}{4}, \frac{1}{2}] \)

(Second interval: \( j = 2 \))

\[
\langle f, 2^j \Psi(2^j t - k) \rangle = \int_0^1 t^2 2 \Psi(4t - 1) \, dt
\]

\[
(0 < 4t - 1 < 1 \Rightarrow \frac{1}{4} < t \leq \frac{1}{2})
\]

\[
= \int_{\frac{1}{4}}^{\frac{1}{2}} t^2 2 \, dt 
\]

\[
= 0.073.
\]

Note that we can continue.

This is the basic idea behind wavelets: We can expand a signal as an average plus finer and finer details.

Note that \( \langle f, 2^j \Psi(2^j t - k) \rangle \rightarrow 0 \) as \( j \rightarrow -\infty \)

(i.e. taking the average over larger and larger intervals)
Multi-resolution analysis

The Hilbert Space $L^2(\mathbb{R})$.

$\mathbb{L}^2(\mathbb{R}) = \{ f : \mathbb{R} \rightarrow \mathbb{C} : \int |f(t)|^2 dt < \infty \}$

(For a proper definition we need the concept of measurability).

The norm of $f$ is defined as:

$$||f|| = \left( \int |f(t)|^2 dt \right)^{1/2}$$

Properties of a norm: $||f|| \geq 0$, $||f|| = 0 \iff f = 0$,

$||cf|| = |c||f||$, $c \in \mathbb{C}$.

$||f+g|| \leq ||f|| + ||g||$

In a Hilbert space, the norm is defined via a scalar product:

$$||f|| = \langle f, f \rangle^{1/2}$$

Recall the Parseval's formula:

$$\int_\mathbb{R} |f(t)|^2 dt = 2\pi \int_\mathbb{R} |f(t)|^2 dt$$

Closed Subspaces and projections

A linear subspace $V \subset \mathbb{L}^2(\mathbb{R})$ is closed if

$$f_n \rightarrow f \text{ in } \mathbb{L}^2(\mathbb{R})$$

implies that $f \in V$.

Example: $\{ f \in \mathbb{L}^2(\mathbb{R}) : f(t) = 0 \text{ when } t < 0 \}$

$$V = \{ f \in \mathbb{L}^2(\mathbb{R}) : |\hat{f}(\omega)| = 0 \text{ when } |\omega| > a \}$$

Definition: A linear operator $P : \mathbb{L}^2(\mathbb{R}) \rightarrow \mathbb{L}^2(\mathbb{R})$

is called a projection if

$$P(f) = P \cdot f \text{ for all } f \in \mathbb{L}^2(\mathbb{R}).$$

Definition: The orthogonal projection of $f$ onto the closed subspace $V$ is the unique $w \in V$ such that

$$||f-w|| = \min_{v \in V} ||f-v||$$

for all $v \in V$.

This projection is denoted by $P_V$.

Here is an example of a subspace $V \subset \mathbb{R}^2$.

It is clear that $P_V(P_V x) = P_V x$.

$\langle x - P_V x, w \rangle = 0$, $\forall w \in V$. 

\[ x \]

\[ P_V x \]

\[ w \]
Bases

A family of functions \( \{\psi_k\}_{k\in\mathbb{Z}} \) is a basis for \( V \subset L^2(\mathbb{R}) \) if any \( f \in V \) can be written (uniquely) as

\[
f = \sum_k c_k \psi_k, \quad c_k \in \mathbb{C}.
\]

Note that if \( \{\psi_k\}_{k}\) is orthogonal, then

\[
\|f\|^2 = \left< \sum_k c_k \psi_k, \sum_k c_k \psi_k \right> = \sum_{k,j} c_k \overline{c_j} \left< \psi_k, \psi_j \right> = \sum_k |c_k|^2.
\]

But for a general basis this is not true.

Def. A Riesz basis for a closed subspace \( V \subset L^2 \) is a basis that satisfies

\[
\|f\|^2 = \sum_k |c_k|^2 \leq B \|f\|^2
\]

for some constants \( A \leq 1 \leq B \).

Note. If \( \tilde{f} \) is an approximation of \( f \)

\[
A \|f - \tilde{f}\|^2 \leq \sum_k |c_k - c_k'|^2 \leq B \|f - \tilde{f}\|^2.
\]

A projection onto \( V \): (orthogonal)

\[
P_V f = \sum_k \langle f, \psi_k \rangle \psi_k
\]

An image of a non-orthogonal projection:

Here \( V = \{(x,0) \in \mathbb{R}^2 \} \).

Biorthogonal bases

A biorthogonal basis is a basis \( \{\psi_k\}_{k} \) of a subspace \( V \) together with a "dual family" \( \{\tilde{\psi}_k\}_{k} \) such that

\[
\langle \psi_k, \tilde{\psi}_n \rangle = \delta_{kn} = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{else} \end{cases}
\]

We let \( P_V f = \sum_k \langle f, \tilde{\psi}_k \rangle \psi_k \).

Note: Not orthogonal projection.
The Haar sampling function

\[ \psi(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases} \]

This can be used for piecewise constant approximation:

\[ f(t) \rightarrow \psi_i(t) = \sum_k s_{ik} \psi(2^i t - k) \]

**Definition** A multi-resolution analysis is a family of closed subspaces \( V_j \subset L^2(\mathbb{R}) \) such that:

1) \( V_j \subset V_{j+1} \), \( j \in \mathbb{Z} \)
2) \( \mathbb{R} = \bigcup_{j \in \mathbb{Z}} V_j \) is dense in \( L^2(\mathbb{R}) \).
3) \( j \in \mathbb{Z} \)
4) \( \bigcap_j V_j = \{0\} \)
5) There is a scaling function \( \phi \in V_0 \)

**Properties of \( \phi \) scaling function**

1) \( \int_{-\infty}^{\infty} \phi(t) \, dt = 1 \)
2) \( k_0 \{ \phi(z-k) \}_{k \in \mathbb{Z}} \) is a basis for \( V_0 \), then we must have \( \{2^{1/2} \psi(2z-k)\}_{k \in \mathbb{Z}} \) is a basis for \( V_1 \). And since \( V_0 \subset V_1 \), \( \psi \in V_1 \) and hence \( \phi(t) = 2 \sum_k h_k \psi(2z-k) \) for some \( \{h_k\} \).

This is called the scaling equation.
Properties of the Scaling Function

3) Let \( H(\omega) = \sum_k h_k e^{-i\omega k} \) and let \( \hat{\phi}(\omega) \) be the Fourier transform of \( \phi \).

Then
\[
\hat{\phi}(\omega) = \sum_k h_k \frac{\phi(2\omega - \omega_k)}{\omega_k},
\]
\[
\sum_{k=\pm \infty} h_k e^{-i\omega k} \hat{\phi}(\frac{\omega}{2}) = H(\frac{\omega}{2}) \hat{\phi}(\frac{\omega}{2}).
\]

Then, by induction
\[
\hat{\phi}(\omega) = \prod_{j \geq 0} H(\frac{\omega}{2^j}).
\]

From \( \hat{\phi}(0) = 1 \) we conclude that \( \sum_k h_k = 1 \).

Example

\( \psi(t) = \text{sinc} t = \frac{\sin \pi t}{\pi t} \Rightarrow \hat{\psi}(\omega) = \eta_{[-1,1]} \).

Wavelet and detail spaces

In a multi-resolution \( \{ V_j \} \), let \( f \) be approximated by \( f_0 \) and \( f_1 \) in \( V_0 \) and \( V_1 \) respectively.

Hence \( f_0 \in V_0 \), \( f_1 \in V_1 \). But also \( f_0 \in V_1 \) \( \Rightarrow \)
\( f_1 - f_0 \in V_1 \).

For the Haar wavelet we have \( \psi(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases} \)

and \( \psi(t) = \frac{1}{\sqrt{2}} \psi_{1,0} - \frac{1}{\sqrt{2}} \psi_{1,1} \).

Def For a MRA, \( \mathcal{V} \) is called a wavelet if
\( W_0 \subset V_1 \) is spanned by \( \{ \psi(\cdot - k) \}_{k \in \mathbb{Z}} \) and
\( V_1 = W_0 + V_0 \), i.e. each \( f \in V_1 \) can be written (uniquely) as
\( f_1 = f_0 + d_0 \) with \( f_0 \in V_0 \), \( d_0 \in W_0 \).

\( W_0 \) is called a detail space.
Properties of the Wavelets

1) \( \int \psi(t) \, dt = 0 \quad (\Rightarrow \hat{\psi}(0) = 0) \)

2) \( \psi(t) \in V_1 \iff \hat{\psi}(\omega) = 2^{-1} \sum_k \hat{\varphi}(2\omega - k) \)

Hence \( \hat{\psi}(\omega) = G(\frac{\omega}{2}) \hat{\varphi}(\omega/2) \)

with \( G(\omega) = \sum_k \hat{\varphi}_k e^{-i\omega k} \),

and because \( \hat{\varphi}(0) = 1 \quad (\hat{\psi}(0) = 0) \)

\( \Rightarrow \hat{\varphi}(0) = \sum_k \hat{\varphi}_k = 0 \).

MRA and wavelet decomposition

We have \( V_1 = V_0 \oplus W_0 \), where \( V_0 \) is spanned by \( \{\varphi(n-k)\}_{n} \) and \( W_0 \) is spanned by \( \{\psi(n-k)\}_{n} \), both of which are supposed to be Riesz bases.

Let \( \psi = \frac{1}{\sqrt{2}} \psi(2^{-1} \cdot n-k) \) and define \( W_j = \text{linear span of } \{\psi_{jk}, k \in \mathbb{Z}\} \).

The set \( \psi_{jk} \) is a Riesz base for \( \mathcal{V} \).

\[ W_j = \{ d_j(t) = \sum_k w_{jk} \psi_j^k(t), \ w_{jk} \in \mathbb{C} \} \]

Let \( J \) be an integer corresponding to highest resolution, or in other words, the finest detail of interest.

Take \( \frac{f}{j} \) be an approximation of \( f \in L^2(\mathbb{R}) \). Then,

\[ \frac{f}{j} \in V_j = W_{j-1} \oplus V_{j-1} \]

\[ = W_{j-1} \oplus W_{j-2} \oplus V_{j-2} \]

\[ = \ldots \]

\[ = W_{j-1} \oplus W_{j-2} \oplus W_{j-3} \oplus \ldots \oplus W_0 + V \]

Then we can write

\[ \frac{f}{j}(t) = \frac{d_{j-1}(t)}{j} + \frac{d_{j-2}(t)}{j} + \ldots + \frac{d_0(t)}{j} \]

\[ = \sum_{j=0}^{j-1} \sum_k w_{jk} \psi_j^k(t) + \sum_k \hat{\varphi}_k \psi_j^k(t) \]

The last sum converges to zero as \( j \to \infty \), because \( \cap \mathcal{V}_0 = \{0\} \).
Also because \( U \subset V \) is dense in \( L^2(\mathbb{R}) \), one can choose \( \frac{f}{J} \rightarrow f \) in \( L^2(\mathbb{R}) \), \( J \in \mathbb{Z} \).

So letting \( J \rightarrow \infty \) we find

\[
f(t) = \sum_{J,k} \alpha_{J,k} \psi_{J,k}(t),
\]

This is the Wavelet decomposition of \( f \).

Example. \( \psi(t) = \text{sinc} \, t = \frac{\sin \pi t}{\pi t} \)

Corresponding to \( \hat{\psi}(w) = \frac{1}{\pi} \text{rect} \left( \frac{w}{\pi} \right) \).

Then \( V_0 \) is the set of bandlimited functions, with cut off \( \pi \), and \( V_j \) the set of bandlimited functions with cut off \( 2^{-j} \pi \).

Here \( \hat{\psi}(w) = \frac{1}{\pi} \text{rect} \left( \frac{w}{\pi} \right) \). \( \psi(t) = \frac{1}{\sqrt{2}} \begin{cases} \frac{\sin \pi t}{\pi t} & \text{if } 0 < t < \pi \\ 0 & \text{otherwise} \end{cases} \)

**Orthogonal Wavelet decomposition**

**General properties of scaling function \( \phi \) and wavelet \( \psi \):**

\[
\begin{cases}
\phi(t) = \sum_k \phi_k \psi(2t-k); \\
\int \phi(t) \, dt = 1
\end{cases}
\]

\[
\begin{cases}
\hat{\phi}(w) = H \left( \frac{w}{2} \right) \hat{\psi} \left( \frac{w}{2} \right), \quad \text{where } H(w) = \sum_k e^{-i2\pi k}, \\
\hat{\phi}(0) = 1 \\
H(0) = 1 \implies \sum_k = 1
\end{cases}
\]

\[
\begin{cases}
\psi(t) = \sum_k \phi_k \psi(2t-k); \\
\int \psi(t) \, dt = 0
\end{cases}
\]

\[
\begin{cases}
\hat{\psi}(w) = G \left( \frac{w}{2} \right) \hat{\phi} \left( \frac{w}{2} \right), \\
G(w) = \sum_k e^{-i2\pi k}w
\end{cases}
\]

**Note.** The decomposition \( \bigoplus_{j=1}^{\infty} V_j \). It is defined by the filter \( G \). However, there is only one way of writing \( V_{j+1} = V_j \oplus W_j \), so that \( \langle V_j, W_j \rangle = 0 \) for all \( v \in V_j \) and \( w \in W_j \).
For an orthonormal system we require

\[ \int_{-\infty}^{\infty} \varphi(t-k) \overline{\varphi(t-n)} \, dt = \begin{cases} 1 & \text{when } n = k, \\ 0 & \text{when } n \neq k. \end{cases} \]

In the scaling equation:

\[ \int_{-\infty}^{\infty} \varphi(t-n) \overline{\varphi(t-m)} \, dt = 2 \sum_{k,k' \in \mathbb{Z}} h_k h_{k'} \int_{-\infty}^{\infty} \varphi(t-2n-k) \overline{\varphi(t-2m-k')} \, dt \]

\[ = 2 \sum_{k,k' \in \mathbb{Z}} h_k h_{k'} \int_{-\infty}^{\infty} \varphi(t-2n-k) \overline{\varphi(t-2m-k')} \, dt = \begin{cases} 1 & \text{if } m = k = 0, \\ 0 & \text{else}. \end{cases} \]

Without loss of generality, we may take \( n = 0 \Rightarrow \)

\[ \int_{-\infty}^{\infty} \varphi(t) \overline{\varphi(t-m)} \, dt = 2 \sum_{k=2m+k' \in \mathbb{Z}} h_k h_{k'} = 2 \sum_{k \in \mathbb{Z}} h_k h_{k+2m, k'} \]

\[ = \sum_{k \in \mathbb{Z}} h_k h_{k+2m} \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{else}. \end{cases} \]

i.e.

\[ \sum_{k} h_k^2 = \frac{1}{2} \quad (\text{Obs! } \sum_{k} h_k = 1). \]

Next we demand that \( \{\varphi(t-k)\}_{k=-\infty}^{\infty} \) is an ON-basis for \( W_0 \):

\[ \int_{-\infty}^{\infty} \varphi(t-k) \overline{\varphi(t-m)} \, dt = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{else}. \end{cases} \]

\[ = \sum_{k \in \mathbb{Z}} g_k g_{k' = 2m} = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{else}. \end{cases} \]

\[ \sum_{k \in \mathbb{Z}} g_k^2 = \frac{1}{2} \quad \text{which implies } \sum_{k \in \mathbb{Z}} g_k = 0. \]

And finally we demand \( \psi \perp W_0 \),

which means that

\[ \int_{-\infty}^{\infty} \varphi(t-k) \overline{\psi(t-n)} \, dt = 0 \quad \text{for all } k, n, \]

which means that

\[ \sum_{m \in \mathbb{Z}} g_{m+2k} g_{m+2n} = 0. \]

The Haar wavelet is an example showing that such systems exist.
Once constructed such a system can be used to approximate \( f \in L^2(\mathbb{R}) \):

\[
\hat{f} = \sum_k \hat{f}_k \quad \text{where} \quad \hat{f}_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt
\]

Proposition: The ON condition implies that:

\[
\left| H(w) \right|^2 + \left| H(w+\pi) \right|^2 = 1 \\
\left| G(w) \right|^2 + \left| G(w+\pi) \right|^2 = 1 \\
H(w) \overline{G(w)} + H(w+\pi) \overline{G(w+\pi)} = 0.
\]

Proof (prove the 1st statement)

\[
H(w) \overline{H(w)} + H(w+\pi) \overline{H(w+\pi)} = \left\{ \begin{array}{c}
\text{assuming } \hat{f}_k \text{ real} \\
\left( \sum_k \hat{f}_k e^{-i\omega k} \right) \left( \sum_k \hat{f}_k e^{i\omega k} \right) + \left( \sum_k \hat{f}_k e^{-i\omega(k+\pi)} \right) \left( \sum_k \hat{f}_k e^{i\omega(k+\pi)} \right)
\end{array} \right.
\]

\[
= \sum_{k,k'} \hat{f}_k \hat{f}_{k'} \left[ (1 + e^{i\pi}) e^{i\omega(k-k')} + 2 \sum_{k+k' = 2m} \hat{f}_k \hat{f}_{k'} e^{-i\omega(k-k')} \right]
\]

\[
= \sum_{k,k'} \hat{f}_k \hat{f}_{k'} \left[ 1 + e^{i\pi} \right] e^{i\omega(k-k')} + 2 \sum_{k+k' = 2m} \hat{f}_k \hat{f}_{k'} e^{-i\omega(k-k')} = \sum_{k,k'} \hat{f}_k \hat{f}_{k'} e^{-i\omega(k-k')} + 2 \sum_{k+k' = 2m} \hat{f}_k \hat{f}_{k'} e^{-i\omega(k-k')}
\]

4) \( \sum_j |\hat{\psi}(2^j w)|^2 = 1 \)

5) \( \sum_j \|\hat{\psi}(2^j(2\pi n+k\pi))\|^2 = 0 \), when \( 2^j \) is odd.

6) \( \sum_j \|\hat{\psi}(2^j(2\pi n+k\pi))\|^2 = 1. \)

Proposition (Conditions on the scaling function and the wavelet)

1) \( \sum |\hat{\phi}(2\pi n+2\pi k)|^2 = 1 \)

2) \( \lim_{\omega \to 0} \hat{\phi}(\omega) = 1 \) (Obvious if \( \hat{\phi} \) continuous at \( \omega = 0 \))

3) \( \hat{\phi}(2\pi) = H(2\pi) \hat{\phi}(\pi), \) \( H \) \( 2\pi \)-periodic

4) \( \sum_j |\hat{\psi}(2^j w)|^2 = 1 \)

5) \( \sum_j \|\hat{\psi}(2^j(2\pi n+k\pi))\|^2 = 0 \), when \( 2^j \) is odd.

6) \( \sum_j \|\hat{\psi}(2^j(2\pi n+k\pi))\|^2 = 1. \)

Proof (1)

\[
\{ r(t) \phi(t-k) \} = \{ 0, k \neq 0 \} \quad \Rightarrow \quad \delta_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(w) e^{i\omega(k-w)} dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(w) e^{i\omega(k-w)} dw
\]

\[
= \sum_n \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\phi}(2\pi n+k\pi)|^2 e^{i\omega(k-2\pi n)} dw
\]

So \( \delta_k \) is the Fourier series of the period function

\[
r(w) = \sum_{n=-\infty}^{\infty} |\hat{\phi}(2\pi n+k\pi)|^2 \quad \text{and} \quad r(w) = 1.
\]
The continuous wavelet transform

We have seen how to write \( f \in L^2(\mathbb{R}) \) as a "wavelet series" (versus "Fourier series"):

\[
\hat{f} = \sum_{j} c_j \psi_j.
\]

Here is a different kind of decomposition:

Take \( \psi \in L^2(\mathbb{R}) \), and assume that

\[
C_\psi = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{\psi}(s)|^2 \, ds < \infty.
\]

If \( \psi \in L^1(\mathbb{R}) \), then \( \hat{\psi}(\xi) \) is continuous, then we must have

\[
\hat{\psi}(0) = \int_{\mathbb{R}} \psi(x) \, dx \rightarrow 0 \quad (\text{wavelet})
\]

The continuity of \( \hat{\psi} \) follows from

\[
|\hat{\psi}(\xi + h) - \hat{\psi}(\xi)| = \int_{\mathbb{R}} (e^{i\xi x} - e^{-ihx}) \hat{f}(x) \, dx
\]

\[
\leq \int_{|x|>1} |e^{i\xi x} - e^{-ihx}| \, dx \leq \int_{|x|>1} |e^{i\xi x} - e^{-ihx}| \, dx = 2|x| \leq 1
\]

Given \( \varepsilon > 0 \), take \( N \) sufficiently large so that \( \int_{|x|>N} |\hat{\psi}(x)| \, dx < \frac{\varepsilon}{4} \), and

\[
|1 - e^{ihx}| < \varepsilon (2\pi |\hat{\psi}(x)|)^{-1} \quad \text{for} \quad |x| < N \quad \text{and} \quad h \in \mathbb{R}.
\]

We then find that

\[
|\hat{\psi}(\xi + h) - \hat{\psi}(\xi)| < \varepsilon \quad \text{for} \quad h \in \mathbb{R}.
\]

Definition

\[
\psi^{ab}(x) = a \psi(\frac{x-b}{a}).
\]

This is a doubly indexed family of wavelets.

Definition:

\[
\langle T^{\text{wavelet}} \hat{f}, \psi^{ab} \rangle = \int_{\mathbb{R}} f(x) \psi(\frac{x-b}{a}) \, dx.
\]

Note: Because \( \|\psi^{ab}\|_2 = 1 \),

\[
\langle T^{\text{wavelet}} \hat{f}, \psi^{ab} \rangle \leq \|f\|_2.
\]

Proposition (The resolution of the identity)

For all \( \hat{f}, \hat{g} \in L^2(\mathbb{R}) \), the following equality holds:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (T^{\text{wavelet}} \hat{f}, T^{\text{wavelet}} \hat{g}) \, da \, db = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(x) \hat{g}(x) \, dx.
\]

This result should be compared with

\[
\int_{\mathbb{R}} g(x) \, dx = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(x) \hat{g}(x) \, dx.
\]
We can write the equality as:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( T^{\text{wav}} f \right)(a,b) \overline{g(x)} \, da \, db \int_{-\infty}^{\infty} \frac{\Psi \left( \frac{x-b}{a} \right)}{al^2} \, dx = \int_{-\infty}^{\infty} g(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( T^{\text{wav}} f \right)(a,b) \overline{\Psi \left( \frac{x-b}{a} \right)} \frac{1}{al^{2+1/2}} \, da \, db \, g(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( T^{\text{wav}} f \right)(a,b) \Psi \left( \frac{x-b}{a} \right) g(x) \, da \, db
\]

So, in a weak sense, this must be equal to \( f \).

**Proof**

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{da \, db}{al^{2+1/2}} \left( T^{\text{wav}} f \right)(a,b) \overline{\left( T^{\text{wav}} g \right)(a,b)}
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int d\xi \, \hat{f}(\xi) \frac{1}{a} \frac{1}{l^{1/2}} \, e^{-ib\xi} \, \overline{\hat{g}(a\xi)} \right] \times
\]

\[
= \frac{1}{2\pi} \int d\xi \, \hat{f}(\xi) \overline{\hat{g}(a\xi)}
\]

\[
= \overline{\hat{f}(a\xi)} \hat{g}(\xi)
\]

Therefore, we can write

\[
\hat{f}(a\xi) = \frac{1}{4\pi^2} \int da \, \int db \, \hat{f}(b) \overline{\hat{g}(a\xi)} = \frac{1}{2\pi} \int \frac{da}{a} \int \hat{f}(b) \overline{\hat{g}(a\xi)} \, db
\]

\[
= \frac{1}{2\pi} \int \frac{da}{a} \int \hat{f}(b) \overline{\hat{g}(a\xi)} \, db \, d\xi = \frac{1}{2\pi} \int \hat{f}(b) \overline{\hat{g}(a\xi)} \, db \, d\xi
\]

\[
= C_{\Psi} \frac{1}{2\pi} \int \hat{f}(b) \overline{\hat{g}(a\xi)} \, db \, d\xi = C_{\Psi} \int f(x) \overline{g(x)} \, db \, d\xi
\]

Note that all integrals of the form \( \int F(x,\xi) \, dx \) are independent of \( \Psi \), if the integral is convergent.
The family \( \{ \psi_{jk} \}_{a,b \in \mathbb{R}} \) is uncountable, but in the discrete wavelet case, we have only countably many \( \psi_{jk} \). How many \( \psi_{jk} \) do we really need?

**Definition** A family \( \{ \psi_j \}_{j \in J} \) in a Hilbert space \( \mathcal{H} \) is called a frame if there exists \( 0 < A \leq B \) such that for all \( f \in \mathcal{H} \),

\[
A \| f \|^2 \leq \sum_j | \langle f, \psi_j \rangle |^2 \leq B \| f \|^2.
\]

\( A \) and \( B \) are called the frame bounds.

A frame is called **tight** if \( A = B \).

**Proposition** If \( \{ \psi_j \}_{j \in J} \) is a tight frame with \( A = 1 \) and \( \| \psi_j \| = 1 \) for all \( j \in J \), then \( \{ \psi_j \} \) is an ON-basis.

**Proof** If \( \langle f, \psi_j \rangle = 0 \) for all \( j \in J \), then \( f \) is orthogonal to all \( \psi_j \), implying \( f = 0 \).

If \( f \neq 0 \) and therefore \( \{ \psi_j \} \) span \( \mathcal{H} \).

Otherwise, let \( \mathcal{H}_0 \) be a span \( \{ \psi_j \}_{j \in J} \) and let \( \mathcal{H}_0^\perp \) be the orthogonal complement of \( \mathcal{H}_0 \) in \( \mathcal{H} \); i.e., \( \mathcal{H}_0 \perp \mathcal{H}_0^\perp \), and \( \mathcal{H}_0 \oplus \mathcal{H}_0^\perp = \mathcal{H} \).

Take \( g \in \mathcal{H}_0 \), \( \| g \| = 1 \). But then \( \langle g, \psi_j \rangle = 0 \) for all \( j \in J \), which contradicts the tightness of the frame.

\[
\| \psi_j \|^2 = \sum_j | \langle \psi_j, \psi_j \rangle |^2 = 1 \quad \text{for each } j \in J.
\]

But \( \| f \|^2 = \| g \|^2 = 1 \implies \sum_j | \langle \psi_j, f \rangle |^2 = 0 \).

So \( \langle \psi_j, \psi_j \rangle = 0 \) if \( j \neq j' \).

If \( f \) is a frame \( \psi_j \) is tight \( \sum_j | \langle \psi_j, \psi_j \rangle |^2 = A \| f \|^2 \)

\( \implies A \langle f, g \rangle = \sum_{j \in J} \langle f, \psi_j \rangle \langle \psi_j, g \rangle \implies f = \frac{1}{A} \sum_{j \in J} \langle f, \psi_j \rangle \psi_j \).

For a frame that is not a ON-basis, there may be many ways \( f \) writing \( f = \sum \psi_j \). The most "economical" way is by aid of the "dual frame" \( \{ \psi_j^* \} \).
Biorthogonal Systems

The figure indicates how $V_j$ and $W_j$ together span $\mathbb{R}^2$, which serves as an image of $V_{j+1}$.

But they are not orthogonal.

Instead, there is a space $\tilde{W}_j$ that is the orthogonal complement to $V_j$. $\tilde{W}_j$ is the detail space of a "dual MRA": $\{\tilde{V}_j\}$; $\tilde{V}_j = \tilde{V}_{j+1}$.

There is a dual scaling function $\tilde{\phi}$ that satisfies a scaling equation:

$$\tilde{\phi}(t) = 2 \sum_{k} \tilde{\phi}(2t-k),$$

and dual mother wavelet that satisfies

$$\tilde{\psi}(t) = \sum_{k} \tilde{\psi}(2t-k).$$

The conditions for biorthogonality are

$$\langle \phi_{jk}, \tilde{\psi}_{jn} \rangle = \delta_{kn}, \quad \langle \phi_{jk}, \tilde{\psi}_{jn} \rangle = \delta_{kn},$$

$$\langle \phi_{jk}, \psi_{jn} \rangle = 0, \quad \langle \tilde{\phi}_{jk}, \psi_{jn} \rangle = 0.$$
Usually the signal is sampled to give a discrete signal \( f_n \).

In the case of the Haar wavelet \( \psi_j,k \) are simply the \( f_n \), but for higher order wavelets the calculation is more involved.

Note that the signal must not contain too high frequencies.

Once the \( \psi_j,k \) are known, we may proceed recursively.

Assume that \( \psi_{j+1,k} \) are known.

We may write \( f_{j+1} = \sum \psi_{j+1,k} \psi_{j+1,k} \) (orthonormality)

and we know that \( f_{j+1} = f_j + \delta_j \).

Hence

\[
f_{j+1}(t) = \sum_{k} \psi_{j+1,k} \psi_{j+1,k}(t) = \sum_{k} \psi_{j,k} \psi_{j,k}(t) + \sum_{k} \psi_{j,k} \psi_{j,k}(t) = f_j(t) + \delta_j(t).
\]

We may compute the scalar product with \( \psi_j \), to get

\[
\sum_{k} \psi_{j+1,k} \psi_{j+1,k} = \sum_{k} \psi_{j+1,k} \psi_{j+1,k} + \sum_{k} \psi_{j+1,k} \psi_{j+1,k} = 0
\]

Now we compute the scalar flux on the LHS:

To this end we use:

\[
\psi_{j,k} = \sqrt{2} \sum_{m} h_m \psi_{j+1,m+2k}
\]

We check (4):

\[
\begin{align*}
\psi_0 &= \psi(t) = \sum_{m} h_m \psi(2t-m) \\
\psi_{j,k} &= \psi(t-k) = \sum_{m} h_m \sqrt{2 \sqrt{2}} \psi(2t-m-2k) \\
&= \sum_{m} h_m \psi_{j+1,2k+m}(t),
\end{align*}
\]

\[
\psi_{j+1,k} \psi_{j+l,k} = \sqrt{2} \sum_{m} h_m \underbrace{\psi_{j+1,k} \psi_{j+1,m+2k}}_{\delta_{k,m+2l}} = \sqrt{2} \delta_{k,l}.
\]

A similar calculation holds for \( \langle f_{j+1}, \psi_{j,k} \rangle \).

In summary:

\[
\psi_{j,k} = \sqrt{2} \sum_{k} h_{k-2j-2l} \psi_{j+1,l} \quad \text{and} \quad \psi_{j,k} = \sqrt{2} \sum_{k} \psi_{j,k} \psi_{j+1,l}.
\]

This may be expressed with a filter bank
The inverse calculation is carried out similarly:

Write

\[ f_{j+1} = \frac{1}{2} \sum_{l} \phi_{j+1, l} \psi_{j+1, l} + \sum_{l} \sum_{m} w_{j+1, l} \psi_{j+1, m} \]

\[ = \sqrt{2} \sum_{l} \sum_{m} \sum_{k} \phi_{j+1, l} \psi_{j+1, m+2k} + \sqrt{2} \sum_{l} \sum_{m} v_{j+1, l} \psi_{j+1, m+2k} \]

and

\[ \psi_{j+1, k} = \langle f_{j+1}, \psi_{j+1, k} \rangle = \sqrt{2} \sum_{l} \sum_{m} \langle \phi_{j+1, l}, \psi_{j+1, m+2k} \rangle \langle \psi_{j+1, m+2k}, \psi_{j+1, k} \rangle \]

\[ = \sqrt{2} \sum_{l} \sum_{m} \langle \phi_{j+1, l}, \psi_{j+1, m+2k} \rangle = \sqrt{2} \sum_{l} \sum_{m} \langle \phi_{j+1, l}, \psi_{j+1, m+2k} \rangle \]

This corresponds to the following filters:

(The factors \( \sqrt{2} \) may be included in the filters.)

At the finest resolution we may then write

\[ f_{j}(t) = \sum_{k} \langle f_{j}, \tilde{\psi}_{j, k} \rangle \psi_{j, k}(t) \]

\[ = \sum_{k} \delta_{j-1, k} \psi_{j-1, k} + \sum_{k} w_{j-1, k} \psi_{j-1, k} \]

and just as in the orthogonal case

\[ f(t) = \sum_{j, k} \langle f, \tilde{\psi}_{j, k} \rangle \psi_{j, k} \]

Note that in this formula we have both \( \tilde{\psi}_{j, k} \) and \( \psi_{j, k} \).