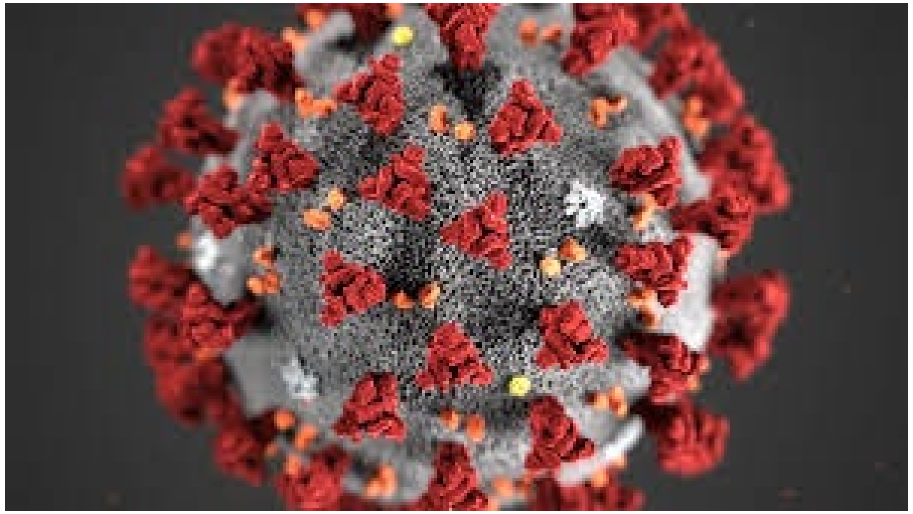


Financial Risk
4-th quarter 2020/21
Lecture 9: Credit risk wrap up

"As an alternative to the traditional 30-year mortgage, we also offer an interest-only mortgage, balloon mortgage, reverse mortgage, upside down mortgage, inside out mortgage, loop-de-loop mortgage, and the spinning double axel mortgage with a triple lutz."

No	S&P	Moody's	Fitch	Meaning and Color
1	AAA	Aaa	AAA	Prime
2	AA+	Aa1	AA+	High Grade
3	AA	Aa2	AA	
4	AA-	Aa3	AA	
5	A+	A1	A+	Upper Medium Grade
6	A	A2	A	
7	A-	A3	A-	
8	BBB+	Baa1	BBB+	Lower Medium Grade
9	BBB	Baa2	BBB	
10	BBB-	Baa3	BBB-	
11	BB+	Ba1	BB+	Non Investment Grade Speculative
12	BB	Ba2	BB	
13	BB-	Ba3	BB-	
14	B+	B1	B+	Highly Speculative
15	B	B2	B	
16	B-	B3	B-	
17	CCC+	Caa1	CCC+	Substantial Risks
18	CCC	Caa2	CCC	Extremely Speculative

Credit rating

Credit risk can be decomposed into:

arrival risk, the risk connected to whether or not a default will happen in a given time-period

timing risk, the risk connected to the uncertainness of the exact time-point of the arrival risk (will not be studied here)

recovery risk. This is the risk connected to the size of the actual loss if default occurs

default dependency risk, the risk that several obligors jointly defaults during some specific time period. This is one of the most crucial risk factors that has to be considered in a credit portfolio framework.

The course focuses default dependency risk for static static credit portfolios where timing risk is ignored

Let L_2 denote the space of all random variables X such that $E\left(X^2\right) < \infty$

Let Z be a random variable and let $L_2(Z) \subseteq L_2$ denote the space of all random variables Y such that $Y = g(Z)$ for some function g and $Y \in L_2$

Note that $E[X]$ is the value μ that minimizes the quantity $E(X - \mu)^2$. Inspired by this, we define the conditional expectation $E[X \mid Z]$ as follows:

For $X \in L_2$, the conditional expectation $E[X \mid Z]$ is the random variable $Y \in L_2(Z)$ that minimizes $E(X - Y)^2$

Properties of conditional expectations

1. If $X \in L_2$, then $E[E[X \mid Z]] = E[X]$
2. If $Y \in L_2(Z)$ then $E[YX \mid Z] = YE[X \mid Z]$

If $X \in L_2$, we define $Var(X|Z)$ as $Var(X|Z) = E[X^2|Z] - E[X|Z]^2$. Then $Var(X) = E[Var(X|Z)] + Var\,(E[X \mid Z])$.

For an event A, we define the conditional probability $P[A| Z]$ as

$$P\,[A| Z] = E[1_A \mid Z]$$

where 1_A is the indicator function for the event A (note that 1_A is a random variable).

An example: if $X \in \{a, b\}$ let $A = \{X = a\}$. Then $P\,[X = a \mid Z] = E[1_{\{X=a\}}|Z]$

The binomial model: m obligors where each obligor can default up to fixed time T, and all have the same constant credit loss ℓ.

Let X_i be a random variable such that

$$X_i = \begin{cases} 1 \text{ if obligor i defaults before time T} \\ 0 \text{ otherwise, i.e. if obligor i survives up to time T} \end{cases}$$

Assume that $X_1, X_2, \ldots X_m$ are i.i.d, that is they are independent with identical distributions, and that $P\left[X_i = 1\right] = p$ so that also $P\left[X_i = 0\right] = 1 - p$.

The total credit loss in the portfolio at time T, called L_m is given by

$$L_m = \sum_{i=1}^{m} X_i \ell = \ell \sum_{i=1}^{m} X_i = \ell N_m \text{ where } N_m = \sum_{i=1}^{m} X_i$$

Thus, N_m is the number of defaults in the portfolio up to time T. Since ℓ is a constant, we have $P\left[L_m = k\ell\right] = P\left[N_m = k\right]$ so it is enough to study the distribution of N_m. It follows from the definition that $N_m \sim Bin(m, p)$

The mixed binomial model

randomizes the default probability and leads to stronger dependence. It works as follows:

Let Z be a random variable (discrete or continuous) and let $p(x) \in [0,1]$ be a function so that also $p(Z)$ is a random variable.

Let $X_1, X_2, ... X_m$ be identically distributed random variables such that $X_i = 1$ if obligor i defaults before time T and $X_i = 0$ otherwise.

Conditional on Z, the random variables $X_1, X_2, ... X_m$ are independent and each X_i has default probability $p(Z)$, that is $P\ [X_i = 1 \mid Z] = p(Z)$

The economic intuition behind this randomizing of the default probability $p(Z)$ is that Z should represent some common background variable affecting all obligors in the portfolio in the same way.

Example 1: A mixed binomial model with $p(Z) = Z$ where Z has a beta distribution, $Z \sim Beta(a, b)$

Example 2: A Logit-normal distribution for p(Z), which means that

$$p(Z) = \frac{1}{1 + \exp(-(\mu + \sigma Z))}$$

where $\sigma > 0$ and μ are parameters, and Z is a random variable which has a standard normal distribution.

Example 3: The mixed binomial model inspired by the Merton model with

$$p(Z) = N\left(\frac{N^{-1}(\bar{p}) - \sqrt{\rho}Z)}{\sqrt{1 - \rho}}\right)$$

where Z is standard normal and $N(x)$ is the distribution function of a standard normal distribution. Furthermore, $\rho \in [0, 1]$ and $\bar{p} = P[Xi = 1]$.

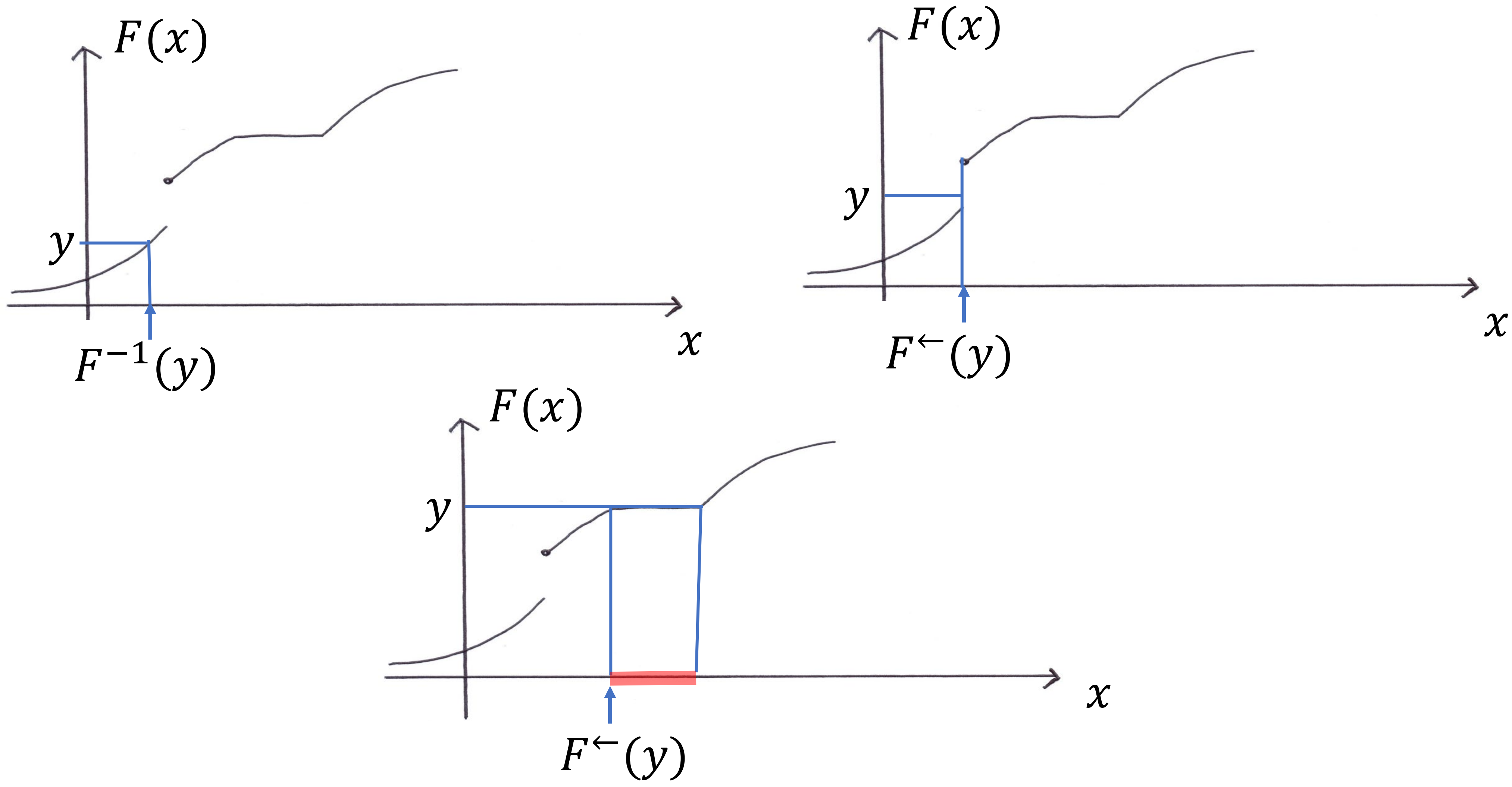
The large portfolio approximation

For large portfolios in a mixed binomial model, the distribution of the fractional number of defaults $\frac{N_m}{m}$ in the portfolio converges to the distribution of the random variable $p(Z)$ as $m \to \infty$, that is for any x ∈ [0, 1] we have

$$P\left(\frac{N_m}{m} \leq x\right) \rightarrow P\left(p(Z) \leq x\right) \quad \text{when} \quad m \rightarrow \infty.$$

The distribution $P\left(p(Z) \leq x\right)$ is called the Large Portfolio Approximation (LPA) to the distribution of N_m / m.

Generalized inverse



Here the inverse could be any value in the red interval. The generalized inverse $F^{\leftarrow}(y)$ is arbitrarily defined to be the leftmost point of the interval

The correlation ρ_X between the default indicators for two obligors in a mixed binomial models is

$$\rho_X = \frac{Ep(Z)^2 - \bar{p}^2}{\bar{p}(1-\bar{p})}$$

In the Merton model ρ_X is the same as the parameter ρ of the model

Monte-Carlo simulation of portfolio credit loss

n = the number of simulations. Choose as large as conveniently possible

For j = 1, 2, . . . , n, repeat the following five steps:

1. Simulate the random variable Z and compute $p(Z) \in [0,1]$.
2. Simulate an i.i.d sequence $U_1, U_2, \ldots, U_m$ with U_i i uniformly distributed on $[0,1]$ and independent of Z.
3. For $i = 1,2,\ldots,m$ define X_i as $X_i = 1$ if $U_i \le p(Z)$ and X_i=0 otherwise
4. If losses are random, simulate $\ell_1(Z), \ell_2(Z), \ldots \ell_m(Z)$
5. Compute $L_j = \sum_{i=1}^{m} X_i \, \ell_i \, (Z)$.

From the simulated sequence $\{L_j, \ j = 1, 2 \ldots n\}$ one obtains the empirical distribution function and can use it to find an estimate of Value-at-Risk etc.