1. QFT: Lecture notes on special topics

1.1 Conventions in Peskin and Schroeder (PS)

See pages xix - xxi in PS.
The flat spacetime metric: \( g_{\mu \nu} = \text{diag}(1, -1, -1, -1) \) and thus \( p \cdot x = p^0 t - p \cdot r \).
The notation \( \eta_{\mu \nu} \) for the Minkowski metric is not used in PS.

1.2 QFT: Introduction and overview of the subject and the course (not in PS)

Why is non-relativistic QM not enough? Or even relativistic QM?

Two key reasons are (to be studied later):
1. Particles can be created and annihilated in scattering processes,
2. Physical processes obey causality.

\[ \text{QFT} = \text{QM (framework)} + \text{field theory (phenomenology)} \]

- The QM framework provides
  - 1) unitarity (conservation of probability)
  - 2) real energies (eigenvalues of the Hamiltonian)
- Nature demands also (no violations ever detected)
  - 3) Poincaré invariance (Lorentz + translations, Lorentz invariance \( \Rightarrow \) causality)
  - 4) stability (there must exist a state, the vacuum, with a lowest possible energy)
  - 5) CPT invariance (see later in the course)
  - 6) spin - statistics theorem: integer spin particles are bosons, half integer spin
    particles are fermions (the latter satisfy the Pauli exclusion principle)
  - 7) conservation of charge (electric and other kinds)
- We want also
  - 8) predictablity, that is renormalisability or finiteness (see later in the course)
  - 9) locality (here this means interacting field theories, see below)
- Note
  - 1), 2), 3) and 4) \( \Rightarrow \) 5) and 6) plus gauge invariance for massless fields of spin 1
    or higher (see below)
  - the above points 1) - 9) \( \Rightarrow \) structure of the standard model of particle physics
gravity is not included here: Einstein’s theory of gravity is an “effective low energy field theory” which is not renormalisable (i.e., not compatible with QFT, more later in the course)

⇒ a consistent quantum gravity theory is needed, e.g., string/M theory. Any low energy field theory that is consistent when coupled to quantum gravity is called **UV complete**. In fact, general relativity appears (as a 2-dim. quantum effect) in string theory as a UV complete generalisation of Einstein’s theory. UV complete theories belong to the **landscape** while non-complete ones belong to the **swampland**. This is hot research subject at the moment (2020).

Quantum methods

- 1. QFT: Field theory (elementary excitations = point particles)
  - i) 2nd quantized field theory (QFT) ⇒ unitarity, stability ($E \geq E_0$): **This course**!
  - ii) path integrals ⇒ Lorentz invariance (Weinberg, QFT, Vol 1, p. 376 - 377)
    * no general proof exists of the equivalence of these two methods
    * for certain restricted Lagrangians there is a proof.

- 2. String theory (elementary excitations = strings, perturbative, see below)

- 3. M-theory (fundamental objects = surfaces, non-perturbative, see below)

Field theories for various spins and their Lagrangians

- Aspects of Lagrangian field theories:
  The Lagrangian $L$ will be very important in this course. In ordinary (Newtonian) mechanics for one particle the Lagrangian is

$$L(x, \dot{x}) := E_{\text{kin}} - E_{\text{pot}} = \frac{1}{2}m\dot{x}^2 - V(x),$$  \hspace{1cm} (1.1)

which can be generalised to many particles, or even infinitely many, by just summing over them $L = \sum_{i=1}^{N} L_i$. This is certainly valid for the kinetic terms in $E_{\text{kin}}$ while the potential $E_{\text{pot}}$ might involve terms with many coordinates and therefore cannot be written as a simple sum like this. This can be further generalised to an integral by viewing the index $i$ as a continuous variable. In field theory the role of this index is then played by the points in space $\mathbb{R}^3$ (or 3-momenta $p$ as we will see later) which for a free massless scalar field becomes (with $x^\mu = (t, \mathbf{r})$ and $\dot{\phi} := \frac{\partial}{\partial t} \phi$)

$$L := E_{\text{kin}} - E_{\text{pot}} = \int d^3x \mathcal{L}(\phi(x), \dot{\phi}(x)) = \int d^3x \left(\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi\right)$$

$$= \int d^3x \left(\frac{1}{2} \dot{\phi}(t, \mathbf{r}) \dot{\phi}(t, \mathbf{r}) - \frac{1}{2} \nabla \phi(t, \mathbf{r}) \cdot \nabla \phi(t, \mathbf{r})\right),$$  \hspace{1cm} (1.2)
where the first term on the last line is the kinetic term $E_{\text{kin}}$ and the second one is part of the potential energy $E_{\text{pot}}$. Using instead a Lorentz covariant language the whole term $\frac{1}{2} \partial_\mu \phi \partial^\mu \phi$ is referred to as the kinetic term. We see also how the overall sign of this term is related to the ”mostly negative” signature used in PS.

The potential energy $E_{\text{pot}}$ in $L$ may also contain mass terms and higher powers of the fields in question. For a real scalar field these are

$$E_{\text{pot}} = \int d^3 x V(\phi) = \int d^3 x \left( \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right). \quad (1.3)$$

The field theory Lagrangian, which is a density, is therefore in this case

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4. \quad (1.4)$$

That this theory cannot contain an odd power of $\phi$ as the highest one or higher powers than four are crucial facts that will be explained later.

Another key aspect of a theory defined in terms of a Lagrangian is the role played by the parameters in it, in the above example $m$ and $\lambda$, and how they must be determined by experiments before the theory has any predictive power. Predictive power must be imposed on any field theory for it to be useful when explaining the outcome of experiments, a fact that puts heavy constrains on the theory. Trying to follow the same logic for general relativity fails as will be clear later in the course.

- Theories familiar from previous courses (at least to some extent).
  - spin 0: Klein-Gordon ($\phi$ real, $\Phi$ complex)
    $$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$
    or $\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi$
  - spin 1/2: Dirac (Weyl, Majorana), derived later!
    $$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m) \psi$$
  - spin 1: Maxwell
    $$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$
    where $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$
  - spin 2: Einstein’s general relativity (GR)
    $$\mathcal{L} = -\frac{1}{16\pi G_N} \sqrt{g} R$$ (the minus sign is due to the signature used in PS)
    where $R = g^{\mu\nu} R_{\mu\nu}$, the Ricci tensor $R_{\mu\nu} = R^\rho_{\mu\nu\rho}$ and the Riemann tensor
    $$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\rho\nu} + \Gamma^\mu_{\eta\sigma} \Gamma^\eta_{\rho\nu} - \Gamma^\mu_{\eta\rho} \Gamma^\eta_{\sigma\nu}$$
    $$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\tau} (\partial_\nu g_{\rho\tau} + \partial_\rho g_{\nu\tau} - \partial_\tau g_{\nu\rho})$$

- Theories introduced in this course (mainly the first three cases)
  - self-interacting neutral spin 0:
    $$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$ where $\phi$ is real
self-interacting charged spin 0: the covariant derivative is \( D_\mu = \partial_\mu + ieA_\mu \)

\[ \mathcal{L} = D_\mu \Phi^* D^\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{2} (\Phi^* \Phi)^2 \] where \( \Phi \in \mathbb{C} \) and \( A_\mu \) is Maxwell

self-interacting spin 1: Yang-Mills theory with gauge group \( G \) (\( i = 1, 2, \ldots, \dim G \))

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \] where \( F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{ijk} A_\mu^i A_\nu^j A_\rho^k \)

* for \( G = SU(2) \) the structure constants \( f^{ijk} = \epsilon^{ijk} \) (the 3d epsilon tensor)

topological spin 1 in 4d: "2nd Chern class" (more later if time permits)

\[ \mathcal{L} = \frac{1}{8\pi^2} \epsilon^{\mu\nu\rho\sigma} Tr F_{\mu\nu} F_{\rho\sigma}, \quad \epsilon^{\mu\nu\rho\sigma} \text{ is the 4d epsilon tensor in Minkowski space} \]

topological spin 1 in 3d: "Chern-Simons theory", important in some condensed matter systems and in string/M theory (more later if time permits)

\[ \mathcal{L} = \frac{k}{4\pi} \epsilon^{\mu\nu\rho} Tr (A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho), \quad (k \in \mathbb{Z}), \quad \epsilon^{\mu\nu\rho} \text{ is the 3d epsilon tensor in Minkowski space} \]

- Physics applications (studied in more advanced courses):
  - Standard model of particle physics: Yang-Mills plus spin 1/2 and 0 fields in 4d
  - Graphene: massless Dirac in 3d plus QED in 4d
  - Topological insulators and FQHE: Chern-Simons plus other fields, all in 3d
  - Superconductors at quantum critical points: conformal in 3d (\( CFT_3 \))
  - Phase transitions: \( CFT_2 \) and \( CFT_3 \)
  - String theory: conformal in 2d (\( CFT_2 \))
  - M-theory: conformal in 3d (\( CFT_3 \)) and in 6d (\( CFT_6 \))

**Spacetime symmetries**

In any dimension the symmetries of a field theory in flat spacetime can be

- Non-relativistic (time is absolute): Galilei
- Relativistic: Poincaré (Lorentz plus translations): **This course**
- Conformal: includes scale invariance (\( CFT_d = \text{conformal field theory in } d \text{ dimensions} \))

**Coupling constant dependence of cross-sections** (and energy levels etc)

A general expansion in coupling constant \( g \) of a cross-section \( \sigma(g) \) is, for \( 0 \leq g < 1 \),

\[ \sigma(g) = \sigma_0 + \sigma_1 g + \sigma_2 g^2 + \ldots + e^{-1/g^2} (\sigma_0^{(1)} + \sigma_1^{(1)} g + \ldots) + e^{-2/g^2} (\sigma_0^{(2)} + \sigma_1^{(2)} g + \ldots) + \ldots \]

- the \( \sigma_n g^n \) terms are called **perturbative** (from perturbation theory: **this course**),
- the terms \( e^{-n/g^2}, n \geq 1 \) are called **non-perturbative** (related to solitons and instantons). These terms do not have a power expansion around \( g = 0 \),
- a very active research area is **resurgence**1 which aims at deriving the non-perturbative terms from the perturbative ones.

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1See, e.g., G. Dunne, ArXiv hep-th/1510.03435.
**Duality**

The modern view on QFT is that the same physics can be described by different field theories where both the field content and the coupling constants may be different:

- Hamiltonian \( H = H_0(\phi) + gH_{\text{int}}(\phi) = H'_0(\phi') + g'H'_{\text{int}}(\phi') \)
  (if evaluated on a physical state, energies are the same since they are measurable!)

- Often the coupling constants are related as \( g' = 1/g \) (strong-weak duality, AdS/CFT)

- The relation between the fields is often very complicated (even non-local)

- The role of elementary excitations and solitons are often interchanged (string theory)

**This course: QFT from second quantised field theory**

- Field theories for spin 0, 1/2, 1, plus comments on Einstein’s theory of gravity (GR)

- Spontaneous symmetry breaking and the Higgs effect

- Perturbation theory

- Feynman graph expansion → scattering amplitudes at "tree" and "1-loop" level

- The physical interpretation of the Lagrangian in field theory: Renormalisation

- Running coupling constants and vacuum polarisation (\( \beta \)-functions if time permits): used to argue that the groups \( U(1) \times SU(2) \times SU(3) \) in the standard model become unified at \( E = 10^{17}\text{GeV} \) since the three coupling constants seem to converge to the same value at this scale.

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2If you are interested, see Polchinski’s review on **Dualities** hep-th/1412.5704.