Problems I on Calculus of Variations, 2020

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February 3, 2020

Problem 1 Consider the problem of minimizing
\[ J(y) = \int_0^1 y(x)^2 y'(x)^2 \, dx, \quad y(0) = 0, \quad y(1) = 1. \]

1. ([2], Problem 1.2) Obtain an upper bound on the minimum using the trial functions \( y_\epsilon(x) = x^\epsilon, \; \epsilon > \frac{1}{4}. \)

2. ([2], Problem 2.4) Find all admissible extremals for the problem.

Problem 2 Find all admissible extremals for
\[ J(y) = \int_0^1 [(y')^2 + 2xy] \, dx, \quad y(0) = 0, \quad y(1) = 1 \]
and a candidate for the minimum. What would change if we had \( y(1) = 0 \) instead? If we had no constraint on \( y(1) \) at all?

Problem 3 Find all (admissible) extremals for
\[ J(y) = \int_0^1 [(y')^2 - y^2] \, dx, \quad y(0) = 0, \quad y(1) = 0. \]
Is there an extremal that gives minimum in \( C^1[0,1] \)? Maximum?

Problem 4 Find the extremals\(^1\) for
\[ J(y) = \int_0^\pi y^2 (1 - (y')^2) \, dx, \quad y(0) = 0, \quad y(\pi) = 0. \]

Problem 5 Find the extremals for
\[ J(y) = \int_0^1 [y^2 + 2xyy'] \, dx, \quad y(0) = 0, \quad y(1) = 1. \]

Problem 6 Find the extremals for the minimal surface problem
\[ J(y) = \int_{-1}^1 y\sqrt{1 + (y')^2} \, dx, \quad y(-1) = y(1) = A. \]
Show that for an extremal to exist it must be \( A \geq 1.5088. \)

\(^1\)We will omit the word admissible from now on.