Partial and ordinary differential equations and systems for chemists

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Chapter 1

Classification of ODEs and PDEs

1.1 Motivation

Why is mathematics in general and differential equations in particular important for chemistry and physics? Mathematics allows us to quantify natural phenomena and make predictions. For example, we might wish to know:

1. How much of each chemical do I need to obtain a certain chemical reaction?

2. How much of the product will I then obtain from this chemical reaction?

3. What temperature do I need for my reaction?

4. In biology and medicine: how much of a particular medication do I need for a particular patient to treat their condition?

Math offers incredible predictive power and can be used to answer questions like these. Chemical reactions generally look like

$$A + B \to C.$$

During this process, the two compounds A and B combine to create C. While this is going on, the *amounts* of A, B, and C are changing over time. Whenever quantities are changing over time, we can describe them using differential equations! Differential equations are all about understanding quantities which change over time. If we can actually *solve* a differential equation, then we can *predict* these quantities at any point in time. Hence - the aforementioned incredible predictive power of mathematics!

1.2 Ordinary differential equations

Even though they are called ordinary, they really are anything but ordinary. Maybe we should call them extraordinary differential equations?

Definition 1.2.1 (ODE)**.** *An "ordinary differential equation" is an equation for an unknown function which depends on one variable.*

Inspired by crime shows, I like to call the unknown function in an ODE the "unsub." We use the variable u to represent the "unsub." Here are some examples:

1. $u'' = u$. Equivalently, we can write this ODE as $u'' - u = 0$. Note here that we don't always write the independent variable. If the independent variable is time, denoted by t, then we could write the same equation as

   $$u''(t) - u(t) = 0.$$

   One reason we can omit the t (no tea no shade) is because the function u depends only on *one* variable. So this shouldn't cause any confusion.

2. Another ODE is:

   $$u^2 = u.$$

   An ODE is an equation for an unknown function of one variable, so it doesn't *necessarily* contain the derivative of the unknown function.

3. Here is an ODE:

   $$t^2 u''(t) + tu'(t) + u(t) = 0.$$

4. Another ODE is:

   $$u'' + \lambda u = 0,$$

   where $\lambda \in \mathbb{C}$ is a constant. An example of this type is:

   $$u'' + 100u = 0.$$

5. The ODE:

   $$u'' = 0$$

   we solved this morning. Let's recall how we did that.

6. We also saw how to obtain all the solutions to the ODE:

   $$au'' + bu' + cu = 0,$$

   Let's recall how to do this here as well.

1.2.1 Classifying ODEs

To *classify* an ODE is a way to give it a name. What's in a name? Would not a rose by any other name smell as sweet? Indeed, a rose by any other name would smell as sweet. However, if we want to search for information about roses, it really helps to know that a rose is called a rose. If we wanted to know about roses, but we didn't know what they are called, how on earth could we do a google search? I suppose you could photograph a rose with your phone and find some app which identifies flowers? To do this, you would at least need to know that a rose is a flower (i.e. you would need to know the word "flower" and what it means). Or, perhaps it would suffice to know that a rose is a plant, and then look for an app which identifies plants. In any case, you need some *key words* to be able to search for information.

It is the same idea with ODEs. I would like to teach you how to give names to the different kinds of ODEs. In this way, if you encounter them in your career as a chemist, you will be able to search for information about them. It does not help to search for information about a second order linear ODE if the equation you have is a fourth order non-linear ODE. What is true for second order linear ODEs does not apply whatsoever to fourth order non-linear ODEs! So, we need to learn how to distinguish between the different types of ODEs.

What is the order?

1. Look in the equation. Look for the highest derivative. This is the *order* of the ODE, and is also called the *degree* of the ODE.

2. Next, look in the equation and see what it is doing to u and its derivatives. In particular, the ODE is *linear* if and only if it is a linear combination of u and its derivatives. So, nothing like
$$u^2$$
is allowed. Similarly
$$u^u$$
is strictly forbidden. If the equation is not linear, then well, we call it *non-linear.*

1.2.2 Examples

Determine the degree of these ODEs, and also whether or not they are linear:

$$y' = 1 + y^2$$

$$y' = ay(b - y)$$

$$tx\dot{x} = 1$$

$$y' = xy$$

$$y' = 1 - y^2$$

$$x^2y' + y = 0$$
$$y''' + 3y'' + 3y' + y = 0$$
$$y'''' + 4y''' + 6y'' + 4y' + y = 0$$

An alternative way to think about differential equations is to use the notion of *an operator.*

Definition 1.2.2. *Every ODE has a canonically associated differential operator, L. To determine the canonically associated ODE operator, L, the ODE should be re-arranged to the form*

$$L(u) = f,$$

where f is an explicitly specified (known) function.

. The idea here is that one takes u and all its derivatives, and shoves them over to the left side of the equation. The right side of the equation is a known function (which could very well be simply 0, the constant $= 0$ function). Each term on the left side of the equation can involve the independent (input) variable of the unknown function, x, as well as the unknown function u, and its derivatives. All of this collected together defines the ODE operator, L. The right side of the equation must not contain either the unknown function, u, nor any of its derivatives. We consider some of the examples above:

1. The ODE $u'' = u$ is of order two. To write the ODE $u'' = u$ using an operator, we re-write it $u'' - u = 0$. The operator is then defined to be in this case

   $$L(u) = u'' - u.$$

   The ODE is

   $$L(u) = 0.$$

   In this case, $f = 0$.

2. The ODE $u^u + u^2 = u$ is an ODE of order *zero.* This is because the unknown function (zero-th order derivative) appears in the ODE, but there are no first or higher order derivatives in the ODE. To write this ODE using an operator, we re-arrange it to

   $$u^u + u^2 - u = 0, \quad L(u) = u^u + u^2 - 2.$$

3. Another ODE is: $u'' + \lambda u = 0$. For this ODE, the operator is $L(u) = u'' + \lambda u$, where λ is a constant.

4. The ODE $u' = 0$ is a first order ODE.

5. What is the order of the ODE, $u = 0$?

These examples motivate another definition.

Definition 1.2.3. *Let L be an ODE operator, with associated ODE*

$$L(u) = f(x).$$

We say that the ODE is homogeneous, *if and only if $f(x) \equiv 0$.*

Why we are bothering to introduce all of these notations and definitions? This is an intelligent thing to be asking at this point. The reason we are doing this is because the aim of this chapter is to *classify* ODEs, and later PDEs. Classifying ODEs and PDEs is a method which gives a precise, technical description of *every ODE and PDE in the universe.* There are different tools and techniques which are useful for solving different classes, or types, of ODEs and PDEs. However, the tools and techniques which can solve one type of ODE or PDE could fail miserably to solve other types of ODEs and PDEs. One would like to avoid such failures. Knowing what kind of ODE or PDE one is trying to solve, by *classifying the equation,* facilitates being able to solve it!

1.3 Classification of ODEs

Recall that a linear function, f, of several variables, x_1, x_2, ..., x_n, can always be expressed as

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} a_j x_j, \quad a_j \in \mathbb{R}\,(\text{or } \mathbb{C}) \text{ for } j = 1, \ldots, n.$$

We shall analogously define *linear* operators.

Definition 1.3.1. *An ODE operator, L, is* linear *if it can be written as a linear combination of the unknown function, u, and its derivatives. A linear ODE operator, L, of order n can always be expressed as*

$$L(u) = \sum_{j=1}^{n} a_j(x) u^{(j)}.$$

Above, u denotes the unknown function, and $u^{(j)}$ denotes the j^{th} derivative of u, where $u^{(0)} = u$. The coefficient functions $a_j(x)$ are specifically given by the ODE. A linear ODE operator L has constant coefficients *if and only if each of the functions $a_j(x)$ is a constant function.*

In the following chapter, we will see a method that will allow us to:

1. determine whether *any* homogeneous, linear ODE with constant coefficients is solvable or it is not solvable;
2. for every solvable such ODE, determine all its solutions.

These techniques are pretty powerful, and surprisingly simple once one gets accustomed to them. Before we get ahead of ourselves, let's consider some examples.

Exercise 1. *Determine in each case the ODE operator, L, and its order. Is L linear or not? Is the ODE homogeneous or not?*

1. $u' + u'' = 0$.
2. $e^u + 1 = 0$
3. $4x^2u''(x) + 12xu'(x) + 3u(x) = 0$.
4. $2tu'4u = 3$
5. $\frac{u'(x)}{u(x)} = e^x$
6. $u'(x) = \frac{x}{u(x)}$
7. $u''(x) = 5$
8. $u'(x) = x^2$
9. $u'(x) + 5u(x) = 2$
10. $u'' = -u$

At this point, one should be able to flip open any book on ODEs and execute the following tasks:

1. identify the ODE operator, L, and its order,
2. determine whether or not L is linear,
3. determine whether or not the ODE is homogeneous.

1.4 Classification of PDEs

Partial differential equations are called so because they involve *partial* derivatives. Partial derivatives are only relevant in the context of functions of several variables.

Definition 1.4.1. *A partial differential equation (PDE) for a function of n real variables is an equation for an unknown function $u : \mathbb{R}^n \to \mathbb{R}$. The* order *of the PDE is the order of the highest partial derivative (or mixed partial derivative) which appears in the equation.*

Here are some examples:

1. For a function $u : \mathbb{R}^2 \to \mathbb{R}$, the equation, $u_{xx} + u_{yy} = 0$. What order is this equation?

2. For a function $u : \mathbb{R}^n \to \mathbb{R}$, the equation,

$$\sum_{j=1}^{n} u_{jj} = \lambda u, \quad \lambda \in \mathbb{R}.$$

   What order is this equation?

3. For $u : \mathbb{R}^3 \to \mathbb{R}$, the equation

$$u_{xyz} - e^x u_x = \sin(yz).$$

   What order is this equation?

We can also express partial differential equations using *operators,* and this will be quite useful.

Definition 1.4.2. *For a PDE of n real variables of order m, the associated PDE operator, L, is defined so that the equation is equivalent to*

$$L(u) = f,$$

where f is an explicitly specified function, with $f : \mathbb{R}^n \to \mathbb{R}$. The PDE is homogeneous if and only if $f \equiv 0$. The PDE is linear *if and only if $L(u)$ has the form*

$$L(u) = \sum_{|\alpha| \leq m} c_\alpha(x) \partial_\alpha u.$$

It has constant coefficients if and only if $c_\alpha(x)$ is constant for all α. Above, α is a multi-index of length at most m, so that if α is a multi-index of length k, then α is of the form $j_1 \ldots j_k$, and

$$\partial_\alpha u = \partial_{j_1} \ldots \partial_{j_k} u,$$

where ∂_{j_1} is the partial derivative in the j_1 coordinate direction.

1.4.1 Classification of second order linear PDEs in two variables

As we have seen in Fourier Analysis, second order linear PDEs in two variables are in fact very important, even if they may seem simple. They are in fact, not that simple, but tractable. For problems in higher dimensions, it may often occur that the "action" is only really occurring in one space direction. Thus, for the laws of physics (and the laws which chemistry obeys as well), we only need to consider one space variable and one time variable: two variables total. Another way in which we are dealing with a three dimensional problem, but the problem can be reduced to a one (space) dimensional problem plus the time variable, is when we are able to separate the different space directions and deal with them individually.

Why is it that so many important PDEs and ODEs (like those with names) are of order two? This is due to *the laws of physics,* so many of which are written with second order PDEs and ODEs. Hence, when we want to understand the behavior of physical (and chemical) systems, we use the laws of physics to describe these systems, and many of these laws are written in the language of PDEs and ODEs. Luckily, many of these laws also happen to be *linear* PDEs. There are some important equations which are *non-linear,* but those are much more difficult to solve. However, a standard way to attack such problems is to *linearize* them, that is to approximate the non-linear problem using a linear problem. It is therefore important to non-linear problems as well to be fluent in the methods used for solving linear PDEs.

To be able to apply the most relevant methods, it helps to be able to specify what type of equation one would like to understand. Imagine trying to search in a library or scholarly database: one needs some *terminology* in order to begin searching! We already have built up some very useful terminology for classifying equations:

1. Is it an ODE or a PDE?
2. What order is the equation?
3. Is the equation homogeneous or inhomogeneous?
4. Is the equation linear or non-linear?
5. If the equation is linear, the does it have constant coefficients or not?

There are a few additional considerations and specifications for second order linear PDEs in two variables. A second order linear PDE in two independent variables, written x and y, can always be written as:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \quad A, \ldots, G \text{ are functions of } x \text{ and } y.$$

A few important examples are:

1. the heat equation, $u_t = u_{xx}$, which has $A = 1$, $E = -1$, and the other capital letters, B, C, D, F, G are all equal to zero. Note that here $y = t$ is the time variable, whereas $x \in \mathbb{R}$ or x in some bounded subset of $\mathbb{R}$ is the spatial variable.
2. The wave equation, $u_{tt} = u_{xx}$. Setting $y = t$, the time variable, what are the values of the coefficients here?
3. Laplace's equation: $u_{xx}+u_{yy} = 0$. Same question: what are the values of the coefficients in this case?

More generally, we have the following classifications:

1. Parabolic: if $B^2 - 4AC = 0$.
2. Hyperbolic: if $B^2 - 4AC > 0$.
3. Elliptic: if $B^2 - 4AC < 0$.

4. None of the above.

If at least one of the coefficients, A, B, C is non-constant, it could happen that none of the above hold. However, if these three coefficients are all constant, clearly one of the three conditions above must hold.

Exercise 2. *Classify the heat equation, wave equation, and Laplace equation.*

Exercise 3. *Classify the following equations:*

1. $u_t = u_{xx} + 2u_x + u$
2. $u_t = u_{xx} + e^{-t}$
3. $u_{xx} + 3u_{xy} + u_{yy} = \sin(x)$
4. $u_{tt} = uu_{xxxx} + e^{-t}$

Exercise 4. *Investigate solutions of the form*

$$u(x,t) = e^{ax+bt}$$

to the equation

$$u_t = u_{xx}.$$

Exercise 5. *Solve:*

$$\frac{\partial u(x,y)}{\partial x} = 0.$$

Exercise 6. *Solve:*

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = 0.$$

Compare with the ODE $u''(t) = 0$. How many solutions are there to the ODE, and what are they? How many solutions are there to the PDE (above)? Describe them.

Chapter 2

Systems of ODEs

Some Chalmers students may recall the Matlab project, Enzymkinetik, which contained the unknown concentrations of four substances each as functions of time. To determine the concentrations of these substances one must therefore solve a system of four first order ODEs. There are many other circumstances in science and engineering which may arise in which we have several functions representing quantities that depend on one another. In a chemical reaction involving 10 different molecules, the quantities of all of these different molecules depend on each other in a specific way. The way in which they depend on each other can be expressed using differential equations! Many of these systems could be *non-linear* which will create some difficulties. However, the first step to understanding non-linear ODEs (and PDEs) is actually to understand their simpler, linear versions. So, we continue to consider linear, constant coefficient homogeneous equations here.

2.1 Systems of ODEs in matrix-vector form

Definition 2.1.1. *A first-order homogeneous system of constant coefficient, linear ODEs, with n unknown functions $u_1, \ldots, u_n$, which each depend on one independent variable, often denoted by t, is an equation*

$$U' = MU, \quad U := \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix},$$

where M is an $n \times n$ matrix.

This equation looks a lot like the single differential equation

$$f' = cf, \quad c \text{ is a constant.}$$

Solutions to that equation are $f(x) = ae^{cx}$ where $a = f(0)$. So, it makes sense to look for a vector version of such a solution for the matrix-vector equation

$$U' = MU.$$

In particular, let's first try a vector of the form

$$U = \begin{bmatrix} c_1 e^{r_1 t} \\ c_2 e^{r_2 t} \\ \vdots \\ c_n e^{r_n t} \end{bmatrix}.$$

Then

$$U' = \begin{bmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & r_n \end{bmatrix} U.$$

Let us call the matrix

$$R = \begin{bmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & r_n \end{bmatrix}.$$

So, the equation is satisfied if and only if

$$U' = RU = MU \iff U = R^{-1}MU.$$

The inverse matrix

$$R^{-1} = \begin{bmatrix} r_1^{-1} & 0 & \dots & 0 \\ 0 & r_2^{-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & r_n^{-1} \end{bmatrix}$$

OBS! The inverse matrix is usually WAY more difficult to calculate. The reason this one is so simple is because the matrix is diagonal. Now, in order to have

$$U = R^{-1}MU \implies R^{-1}M = \text{ the identity matrix,}$$

which has ones on the diagonal and zeros everywhere else. By definition of inverse matrix, then

$$M = R.$$

So, a vector of this form is only a solution when the matrix M is a diagonal matrix. When M is a diagonal matrix, then the system of equation looks like:

$$u_1' = r_1 u_1$$

$$u_2' = r_2 u_2$$

$$\vdots$$

$$u_n' = r_n u_n.$$

In particular, these are just n equations that have nothing to do with each other. It's not super interesting, and we know how to solve these. What about when M is not of this form?

For general M, we will look for solutions of the form

$$U = Ve^{\lambda t}, \quad V \in \mathbb{C}^n, \quad \lambda \in \mathbb{C}.$$

Then

$$U' = V\lambda e^{\lambda t} = MU \iff V\lambda e^{\lambda t} = MVe^{\lambda t}.$$

Dividing both sides of the last equality by $e^{\lambda t}$, we see that a function $U = Ve^{\lambda t}$ is a solution to the equation if and only if

$$MV = \lambda V.$$

This holds if and only if V is an eigenvector for the matrix M, and λ is the corresponding eigenvalue. Note that for U of this type,

$$U(0) = V.$$

Theorem 2.1.2. *Let M be an $n \times n$ matrix. Then the eigenvalues of M are the roots of its characteristic polynomial*

$$p(x) = \det(M - xI),$$

where I is the $n \times n$ identity matrix. There are precisely n eigenvalues, counting multiplicity, denoted by λ_1, λ_2, $\ldots \lambda_n$, with

$$p(x) = a \prod_{j=1}^{n} (x - \lambda_j),$$

for a constant $a \in \mathbb{C}$, with each of $\lambda_j \in \mathbb{C}$ for $j = 1, \ldots, n$. The eigenvalues which occur precisely once are simple. *Each eigenvalue has one or more corresponding eigenvectors, so that for an eigenvalue λ, there is at least one vector $V \in \mathbb{C}^n$ with*

$$MV = \lambda V.$$

Exercise 7. *Show that if M has real valued matrix entries and $\lambda \in \mathbb{C}$ is an eigenvalue of M, then $\overline{\lambda}$ is also.*

The eigenvalues of the $n \times n$ matrix, M, are the roots of its *characteristic polynomial,*

$$p(x) = \det(M - xI).$$

Above, I is the $n \times n$ identity matrix, which has ones along the diagonal and zeros everywhere else. The polynomial $p(x)$ is a polynomial of degree n. By the Fundamental Theorem of Algebra, the characteristic polynomial factors over $\mathbb{C}$, so that

$$p(x) = a \prod_{j=1}^{n} (x - \lambda_j), \quad \lambda_1, \ldots, \lambda_n \in \mathbb{C}.$$

The numbers λ_j don't need to be different, they could all be the same. For example, the matrix

$$M = I \implies p(x) = \det(I - xI) = (1-x)^n = (-1)^n \prod_{j=1}^{n} (x-1) \implies \lambda_1 = \lambda_2 = \ldots = \lambda_n = 1.$$

The number of times a specific number appears in the list $\lambda_1, \ldots, \lambda_n$ is its *algebraic multiplicity.*

Actually finding the eigenvalues of a matrix is pretty annoying, and it becomes more and more annoying the larger the matrix is. Fortunately, matrices come up all over the place; did you know that Google is fundamentally based on really large matrices? So, the good news is that one must simply stick the matrix into a computer program or a sophisticated calculator, and technology does the annoying work. The skills required by the human are thus reduced to the following tasks:

1. Do the individual equations each only have one unknown function in them? If so, then we can solve all the equations individually.

2. If not, then put the system of first order equations into matrix-vector form, defining M and U as above according to your equations. (If the matrix is diagonal then the individual equations only have one unknown function in each, so return to step one).

3. Ask a computer to find the eigenvectors and eigenvalues of the matrix.

4. If there are no initial conditions, then any

   $$Ve^{\lambda t},$$

   such that V is an eigenvector with eigenvalue λ is a solution.

5. If the initial condition, $U(0)$ is specified, then there is a solution if and only if there is an eigenvector V asuch that

   $$U(0) = V$$

   If so, then $U(t) = Ve^{\lambda t}$ is the solution, where λ is the eigenvalue for V.

A more sophisticated way to explain the last condition above is that we are checking to see if the initial data $U(0)$ is in one of the eigenspaces. For an eigenvalue λ, the eigenspace associated to λ is the span of all the eigenvectors which have eigenvalue equal to λ.

2.1.1 Turning a higher order ODE into a system of first order ODEs

Another way to obtain a first-order homogeneous system of constant coefficient, linear ODEs is to start with a higher order ODE. For example, consider the equation

$$u''' + 2u'' - u' + 3u = 0.$$

Exercise 8. *Classify the above equation.*

This is a linear, homogeneous ODE with constant coefficients. We can use the same matrix-system technique to solve this higher order equation in the following way. Let $u_0 = u$, $u_1 = u'$, $u_2 = u''$. We can write the ODE as

$$u_0' = u_1, \quad u_1' = u_2, \quad u_2' = -2u_2 + u_1 + 3u_0.$$

Let

$$U = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}.$$

The equation is now

$$U' = \begin{bmatrix} u_1 \\ u_2 \\ 3u_0 + u_1 - 2u_2 \end{bmatrix} = MU,$$

where

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -2 \end{bmatrix}.$$

To solve the system:

1. Begin by classifying the ODE. Make sure it is linear, has constant coefficients, and is homogeneous. Assume it has degree n.

2. Define

$$U = \begin{bmatrix} u_0 \\ u_1 \\ \cdots \\ u_{n-1} \end{bmatrix}$$

   with

$$u_0 = u, \quad u_1 = u', \dots, u_{n-1} = u^{(n-1)},$$

   where u is the unknown function we seek to satisfy the ODE.

3. Look at the ODE. Re-arrange it to look like:

$$u^{(n)} = \dots,$$

   where the right side contains u and its derivatives of order *less than* n.

4. Remember that, the way we've defined things,

$$
\begin{aligned}
u_0' &= u_1 \\
u_1' &= u_2 \\
u_2' &= u_3 \\
&\vdots \\
u_{(n-1)}' = u^{(n)} &= \ldots \text{ terms with } u_0,\, u_1, \text{ and up to } u_{n-1}.
\end{aligned}
\tag{2.1.1}
$$

Collect these equations to define a matrix M such that the ODE is equal to

$$U' = \begin{bmatrix} u_0' \\ u_1' \\ \cdots \\ u_{n-1}' \end{bmatrix} = MU = M \begin{bmatrix} u_0 \\ u_1 \\ \cdots \\ u_{n-1} \end{bmatrix}.$$

5. Use software to find the eigenvalues and eigenvectors of M.

Exercise 9. *Put the following systems of ODEs into matrix form:*

1. $u_1' = 4u_1 + 7u_2$ *und* $u_2' = -2u_1 - 5u_2$

2. $u_1' = 3u_2 + u_3,\ u_2' = u_1 + u_2 + u_3,\ u_3' = 0.$

Put the following higher order ODEs into matrix form:

1. $2y'' - 5y' + y = 0$

2. $y^{(4)} - 3y'' + y' + 8y = 0$

Tip: In order for a system of ODEs to be solvable, one requires the same number of *linearly independent* equations as the number of unknown functions. The reason for this is that to use a matrix and its eigenvalues, one needs the matrix to be square, that is the same number of columns as rows. There is no such thing as the eigenvalue or eigenvector of a non-square matrix. Once the system of ODEs has been put into matrix form, as

$$U' = MU,$$

then one solves for the eigenvalues of M and corresponding eigenvectors.

2.1.2 Summary

For a system of first order, linear, homogeneous ODES (whether it came from a higher order ODE or not), write it as

$$U' = MU,$$

where M is a matrix.

1. Is M an $n \times n$ matrix for some $n \in \mathbb{N}$? If the answer is *yes,* then we can continue to find the solutions. If the answer is no, then we stop.

2. In case M is an $n \times n$ matrix, use some technological assistance to find all its eigenvalues and corresponding eigenvectors.

3. General solutions, without any specified data, are all functions of the form

$$U(t) = Ve^{\lambda t},$$

such that V is in the eigenspace of λ, and λ is an eigenvalue of M.

4. To find a *particular solution,* we need to know the initial data,

$$U(0).$$

There exists a particular solution if and only if for some eigenvalue λ, $U(0)$ is contained in the eigenspace of λ. (Remember the eigenspace is the span if the eigenvectors who have eigenvalue equal to λ.)

2.2 The magical Laplace transform

Later in this course, we will learn about something known as the *Laplace transform.* It is defined for functions which do not grow super-exponentially.

Definition 2.2.1. *Assume that*

$$f(t) = 0 \quad \forall t < 0, \tag{2.2.1}$$

and that there exists $a, C > 0$ *such that*

$$|f(t)| \leq Ce^{at} \quad \forall t \geq 0. \tag{2.2.2}$$

Then for we define for $z \in \mathbb{C}$ *with* $\Re(z) > a$ *the Laplace transform of* f *at the point* z *to be*

$$\mathfrak{L}f(z) = \int_0^\infty f(t)e^{-zt}dt.$$

Exercise 1. *Show that if f is continuous and piecewise $\mathcal{C}^1$ on $[0,\infty)$, and f' satisfies* (2.2.2) *and* (2.2.1), *then*

$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$

(Hint: integrate by parts!)

The Laplace transform can be used to solve any linear, constant coefficient ODE, whether it is homogeneous or not! This is super amazing. Time permitting, we will learn how to do this here.

Proposition 2.2.2. *Assume that everything is defined, then*

$$\mathfrak{L}(f^{(k)})(z) = z^k\mathfrak{L}f(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j-1}.$$

Proof: Well clearly we should do a proof by induction! Check the base case first:

$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$

Here $k = 1$ and the sum has only one term with $j = k = 1$. It works. Now we assume the above formula holds and we show it for $k+1$. We compute

$$\mathfrak{L}(f^{(k+1)})(z) = \mathfrak{L}((f^{(k)})')(z) = z\mathfrak{L}(f^{(k)})(z) - f^{(k)}(0).$$

By induction this is

$$z\left(z^k\mathfrak{L}f(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j-1}\right) - f^{(k)}(0).$$

This is

$$z^{k+1}\mathfrak{L}f(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j} - f^{(k)}(0).$$

Let us change our sum: let $j+1 = l$. Then our sum is

$$\sum_{l=2}^{k+1} f^{k-(l-1)}(0)z^{l-1} = \sum_{l=2}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

Observe that

$$f^{(k)}(0) = f^{k+1-1}(0)z^{1-1}.$$

Hence

$$-\sum_{j=1}^{k} f^{(k-j)}(0)z^{j} - f^{(k)}(0) = -\sum_{l=1}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

So, we have computed

$$\mathfrak{L}(f^{(k+1)})(z) = z^{k+1}\mathfrak{L}f(z) - \sum_{l=1}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

That is the formula for $k+1$, which is what we needed to obtain.