Exercises on Linear systems of ODE with constant coefficientsin considered on the Lecture 4 and as a homework after and before the lecture for the course ODE and modeling MMG511/TMV162

Find general solutions to following ODEs and sketch phase portraits for systems in plane:

792. $\begin{cases} x' = & 2x + y \\ y' = & -x + 4y \end{cases}$ **- Homework**

853. $r' = Ar$ with $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$, **given as an exercise in the class.**

854. $r' = Ar$ with $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$, **- solved in the class, complex eigenvalues**

856. $r' = Ar$ with $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & -2 \\ 1 & 5 & -3 \end{bmatrix}$, **- Homework**

859. $r' = Ar$ with $A = \begin{bmatrix} 3 & -3 & 1 \\ 3 & -2 & 2 \\ -1 & 2 & 0 \end{bmatrix}$, **- solved in the class, complex eigenvalues**

862. $r' = Ar$ with $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, **- can be solved in the class**

863. $r' = Ar$ with $A = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & 0 & 3 \end{bmatrix}$, **- Homework**

864. $r' = Ar$ with $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 2 & -3 \end{bmatrix}$, **solved in the class: a complicated case when eigenvectors must be chosen in a clever way**

Answers and solutions.

Theoretical background. We use the formula

$$x(t) = e^{At}x_0 = \sum_{j=1}^{s}\left(\left[\sum_{k=0}^{m_j-1}(A-\lambda_j I)^k \frac{t^k}{k!}\right] x^{0,j} e^{\lambda_j t}\right)$$

for solutions with initial data

$$x(0) = x_0 = \sum_{j=1}^{s} x^{0,j}$$

with $x^{0,j} \in E(\lambda_j, A)$ - components of x_0 in the generalized eigenspaces $E(\lambda_j, A)$ $= \ker(A-\lambda_j)^{m_j}$ of the matrix A. Here s is the number of distinct eigenvalues λ_j to A and m_j is the algebraic multiplicity of the eigenvalue λ_j. We point out that $\mathbb{C}^n = E(\lambda_1, A) \oplus E(\lambda_2, A) \oplus ... \oplus E(\lambda_s, A)$.

General solution can be expressed more explicitely by finding a basis of $\mathbb{C}^n$ in terms of eigenvectors v_j and generalized eigenvectors $v_j^{(k)}$ $k = 1, ...l \le m_j - 1$ corresponding to all distinct eigenvalues to A: λ_j, $j = 1, ...s$, so that components $x^{0,j}$ of x_0 on to the generalized eigenspaces are expressed in the form

$$x^{0,j} = ...C_p v_j + C_{p+1} v_j^{(1)} + C_{p+2} v_j^{(2)} ...$$

including all linearly independent eigenvectors corresponding to λ_j (it might be several eigenvectors v_j corresponding to one λ_j) and corresponding linearly independent generalized eigenvectors for example calculated as it is suggested below.

Eigenvectors and generalized eigenvectors is convenient to calculate as a chain of vectors satisfying the following recursive chain of equations

$$\begin{aligned}
(A-\lambda_j I)\, v_j &= 0, \\
(A-\lambda_j I)\, v_j^{0,1} &= v_j \\
(A-\lambda_j I)\, v_j^{0,2} &= v_j^{0,1} \\
& \quad e.t.c. \\
(A-\lambda_j I)\, v_j^{0,n_j-1} &= v_j^{0,n_j-2}
\end{aligned}$$

It is not always straightforward to run this algorithm from the top downward, depending on the matrix and the choice of the eigenvectors that is not unique. Sometimes the only way is to find a generalised eigenvector v_j^{0,n_j-1} using the definition solving the equation: $(A-\lambda_j I)^{n_j} v_j^{0,n_j-1} = 0$ for n_j such that $(A-\lambda_j I)^{n_j-1} v_j^{0,n_j-1} \ne 0$. After that one can apply the same algorithm in the upward direction. Substituting this expression for x_0 in to the general formula above and carrying out all matrix-matrix and matrix-vector, multiplications one gets a general solution. Keep in mind that $(A-\lambda_j I)\, v_j = 0$ and $(A-\lambda_j I)^2 v_j^{0,1} = 0$ e.t.c., so many terms in the expression $\left[\sum_{k=0}^{m_j-1}(A-\lambda_j I)^k \frac{t^k}{k!}\right] x^{0,j}$ for $x^{0,j} = C_p v_j + C_{p+1} v_j^{(1)} + C_{p+2} v_j^{(2)} + ...$ are zero.

792. Answer. $x = (C_1 + C_2 t) e^{3t}$; $y = (C_1 + C_2 + C_2 t) e^{3t}$

853. Answer. $r = \begin{bmatrix} 2C_1 e^{-t} + C_2 e^{-t}(2t+2) \\ 2C_1 e^{-t} + C_2 e^{-t}(2t+1) \end{bmatrix}$

Solution: $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

characteristic polynomial: $\lambda^2 + 2\lambda + 1 = 0$ has a double eigenvalue: $\lambda = -1$, and one linearly independent eigenvector: $v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Generalized eigenvector $v^{(1)} = \begin{bmatrix} x \\ y \end{bmatrix}$ satisfies the equation

$\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Longrightarrow 2x - 2y = 2;\ y = 1,\ x = 2;\ v^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Observe that v and $v^{(1)}$ are linearly independent (not parallel in the plane).

Therefore any initial data r_0 can be represented as $r_0 = C_1 v + C_2 v^{(1)}$ and solution to I.V.P. with initial data r_0 will be

$$\begin{aligned} r(t) &= e^{At} r_0 = C_1 e^{\lambda t} v + [I + (A - \lambda I)t] e^{\lambda t} C_2 v^{(1)} \\ &= C_1 e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{-t} C_2 \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \right) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \\ &= C_1 e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + C_2 \left(e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right) = C_1 e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 2t+2 \\ 2t+1 \end{bmatrix} \\ &= \end{aligned}$$

$$r(t) = \begin{bmatrix} 2C_1 e^{-t} + C_2 e^{-t}(2t+2) \\ 2C_1 e^{-t} + C_2 e^{-t}(2t+1) \end{bmatrix}$$

: ■

854. Answer. $r = C_1 e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$

Solution. $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$, characteristic polynomial: $\lambda^2 - 2\lambda + 5 = 0$;

eigenvectors:$v_1 = \left\{ \begin{bmatrix} 1 \\ 1+i \end{bmatrix} \right\} \leftrightarrow \lambda_1 = 1 - 2i$,and $v_2 = \left\{ \begin{bmatrix} 1 \\ 1-i \end{bmatrix} \right\} \leftrightarrow \lambda_2 = 1 + 2i$.

A complex solution is $x^*(t) = e^{(1-2i)t} \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$.

Two linearly independent solutions can be chosen as real and imaginary part of $x^*(t)$ and can be used for representing a general solution as $x(t) = C_1 \operatorname{Re}[x^*(t)] + C_2 \operatorname{Im}[x^*(t)]$.

$$e^{(1-2i)t} \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = e^t (\cos 2t - i \sin 2t) \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = e^t \begin{bmatrix} \cos 2t - i \sin 2t \\ (1+i)\cos 2t + (1-i)\sin 2t \end{bmatrix}$$

$$= e^t \begin{bmatrix} \cos 2t - i \sin 2t \\ \cos 2t + \sin 2t + i(\cos 2t - \sin 2t) \end{bmatrix} = e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} - i\, e^t \begin{bmatrix} \sin 2t \\ (\sin 2t - \cos 2t) \end{bmatrix}$$

Answer follows as linear combination of real and imaginary parts:

$x(t) = C_1 \operatorname{Re}[x^*(t)] + C_2 \operatorname{Im}[x^*(t)]$. ■

856. Answer. $r = C_1 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

Hints to finding complex eigenvectors.

858. Answer $r = C_1 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} \cos 2t \\ -\sin 2t \\ \cos 2t \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} \sin 2t \\ \cos 2t \\ \sin 2t \end{bmatrix}$

Two linearly independent solutions can be chosen as above, as real and imaginary part of one of the complex conjugate complex solutions $x^*(t)$ corresponding to a complex eigevalue and can be used for representing a general solution. A complication in the present case is to find complex eigenvectors satisfying a homogeneous system of three equations.

$A = \begin{bmatrix} -3 & 2 & 2 \\ -3 & -1 & 1 \\ -1 & 2 & 0 \end{bmatrix}$, characteristic polynomial: $p(\lambda) = \lambda^3 + 4\lambda^2 + 9\lambda + 10$,

roots: $\lambda_1 = -2$, $\lambda_2 = -1 - 2i$, $\lambda_3 = \overline{\lambda_2} = -1 + 2i$. The real root one can guess, two other are found from a quadratic equation.

An eigenvector corresponding to the eigenvalue $\lambda_2 = -1 - 2i$ satisfies homogeneous system with matrix $A - \lambda_2 I$:

$A - \lambda_2 I = \begin{bmatrix} -3 - (-1 - 2i) & 2 & 2 \\ -3 & -1 - (-1 - 2i) & 1 \\ -1 & 2 & -(-1 - 2i) \end{bmatrix} = \begin{bmatrix} -2 + 2i & 2 & 2 \\ -3 & 2i & 1 \\ -1 & 2 & 1 + 2i \end{bmatrix}$

Change order of rows and multiply the first row by -1:

$\begin{bmatrix} 1 & -2 & -1 - 2i \\ 1 - i & -1 & -1 \\ 3 & -2i & -1 \end{bmatrix}$,

Multiply the second row by the conjugate $1 + i$ of it's first non-zero element $1 - i$ to simplify Gauss elimination and use that $(1 + i)(1 - i) = 1 + 1 = 2$.

In general for $z = a + ib$ and it's complex conjugate $\overline{z} = a - ib$

$$z\,\overline{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$$

$\rightarrow \begin{bmatrix} 1 & -2 & -1 - 2i \\ 2 & -1 - i & -1 - i \\ 3 & -2i & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 - 2i \\ 0 & 3 - i & 1 + 3i \\ 0 & 6 - 2i & 2 + 6i \end{bmatrix} \rightarrow$

$\begin{bmatrix} 1 & -2 & -1 - 2i \\ 0 & 3 - i & 1 + 3i \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$

Multiply the second row by the conjugate $3 + i$ of it's first non-zero element $3 - i$ an use that $(3 + i)(3 - i) = 9 + 1 = 10$:

$\rightarrow \begin{bmatrix} 1 & -2 & -1 - 2i \\ 0 & (3 - i)(3 + i) & (1 + 3i)(3 + i) \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 - 2i \\ 0 & 10 & 10i \\ 0 & 0 & 0 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & -2 & -1-2i \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix}$$

Choosing components in v_2 as $x_3 = 1$ we get $x_2 = -i$, and $x_1 = 1$ and

$$v_2 = \begin{matrix} 1 \\ -i \\ 1 \end{matrix}.$$

The second complex eigenvector corresponding to the conjugate eigenvalue λ_3 is complex conjugate to v_2 because the matrix A is real: $v_2 = \overline{v_3}$ and $\lambda_2 = \overline{\lambda_3}$ are congugate.

■

859. $x' = Ax$. $A = \begin{bmatrix} 3 & -3 & 1 \\ 3 & -2 & 2 \\ -1 & 2 & 0 \end{bmatrix}$ was considered at the lecture.

Answer. $r = C_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + C_2 e^t \begin{bmatrix} \cos t - \sin t \\ \cos t \\ \sin t \end{bmatrix} + C_3 e^t \begin{bmatrix} \cos t + \sin t \\ \sin t \\ -\cos t \end{bmatrix}$

Solution.

The characteristic polynomial is : $\lambda^3 - \lambda^2 + 2 = (\lambda+1)\left(\lambda^2 - 2\lambda + 2\right) = 0$.

Eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ corresponding to $\lambda_1 = -1$.satisfies the equation $(A+I)\, v_1 = 0$

$(A+I) = \begin{bmatrix} 4 & -3 & 1 \\ 3 & -1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$, Gaussian elimination gives: $\begin{bmatrix} 4 & -3 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Hints to finding complex eigenvectors.

Eigenvectors to the eigenvalue $\lambda_2 = 1 - i$ are found from the homogeneous system of equations with the following matrix.

Change places for the first and the last rows and multiply the new first row by -1.

$$\begin{bmatrix} 2+i & -3 & 1 \\ 3 & -3+i & 2 \\ -1 & 2 & -1+i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1-i \\ 3 & -3+i & 2 \\ 2+i & -3 & 1 \end{bmatrix}$$

Multiply the last row by the conjugate of the first element to simplify Gauss elimination: $\rightarrow \begin{bmatrix} 1 & -2 & 1-i \\ 3 & -3+i & 2 \\ (2+i)(2-i) & -3(2-i) & (2-i) \end{bmatrix} = \rightarrow \begin{bmatrix} 1 & -2 & 1-i \\ 3 & -3+i & 2 \\ 5 & -6+3i & 2-i \end{bmatrix} \rightarrow$

$$\begin{bmatrix} 1 & -2 & 1-i \\ 0 & 3+i & -1+3i \\ 0 & 4+3i & -3+4i \end{bmatrix}$$

Multiply rows 2 and 3 by conjugates of pivot elements in each row to simplify Gauss elimination:

$$\to \begin{bmatrix} 1 & -2 & 1-i \\ 0 & (3+i)(3-i) & (-1+3i)(3-i) \\ 0 & (4+3i)(4-3i) & (-3+4i)(4-3i) \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1-i \\ 0 & 10 & 10i \\ 0 & 25 & 25i \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1-i \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix}$$

Chose $x_3 = 1$, $x_2 = -i$, $x_1 = -1-i$.

The second eigenvector corresponding to the conjugate eigenvalue is complex conjugate because the matrix A is real: $v_2 = \overline{v_3}$ and $\lambda_2 = \overline{\lambda_3}$ are congugate.

Eigenvectors and eigenvalues are: $v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \leftrightarrow \lambda_1 = -1, v_2 = \begin{bmatrix} -1-i \\ -i \\ 1 \end{bmatrix} \leftrightarrow$ $\lambda_2 = 1-i$, $v_3 = \begin{bmatrix} -1+i \\ i \\ 1 \end{bmatrix} \leftrightarrow \lambda_3 = 1+i$,

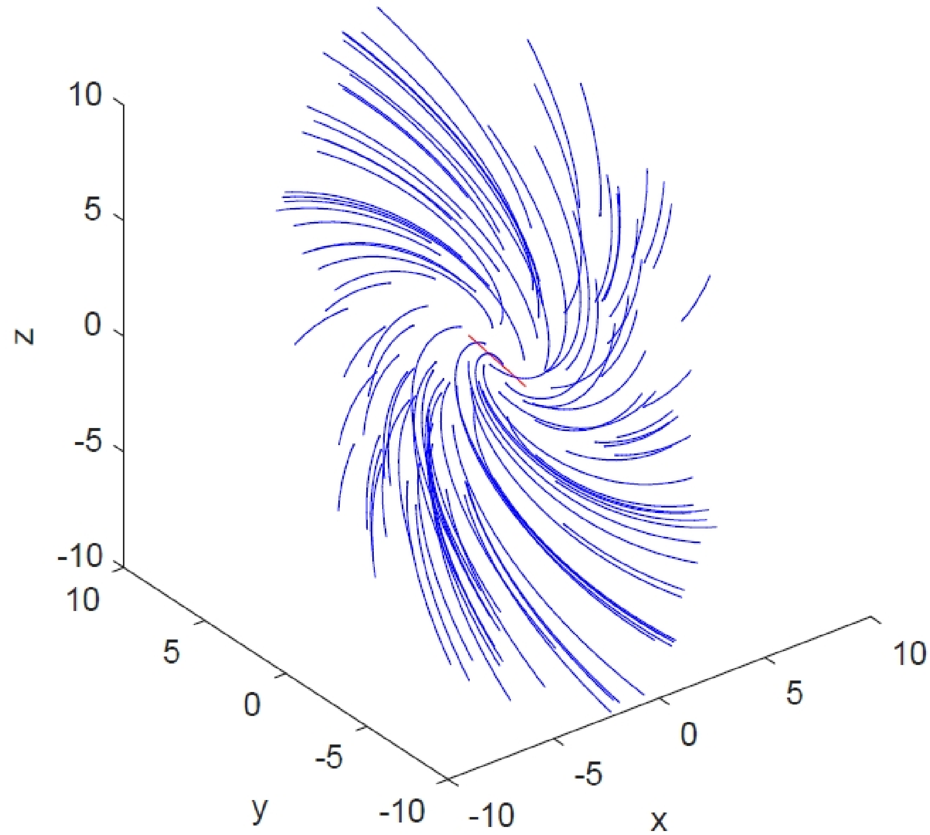
Eigenvalues are all simple, therefore eigenvectors are linearly independent and general complex solutions are expressed as $x(t) = \sum_{k=1}^{3} C_k e^{\lambda_k t} v_k$. If we look for general real solutions that is natural for a real matrix A, we can use solution real and imaginary parts of the complex solution $x^*(t) = v_2 e^{\lambda_2 t}$ as two linearly independent real solutions to the ODE in addition to $e^{\lambda_1 t} v_1$.

$$x^*(t) = e^{(1-i)t} \begin{bmatrix} -1-i \\ -i \\ 1 \end{bmatrix} = e^t(\cos t - i \sin t) \begin{bmatrix} -1-i \\ -i \\ 1 \end{bmatrix} = e^t \begin{bmatrix} -(1+i)\cos t - (1-i)\sin t \\ -i\cos t - \sin t \\ \cos t - i \sin t \end{bmatrix}$$

$$= e^t \begin{bmatrix} -(1)\cos t - (1)\sin t - (i)\cos t - (-i)\sin t \\ -\sin t - i\cos t \\ \cos t - i \sin t \end{bmatrix} = e^t \begin{bmatrix} -\cos t - \sin t \\ -\sin t \\ \cos t \end{bmatrix} + ie^t \begin{bmatrix} -\cos t + \sin t \\ -\cos t \\ -\sin t \end{bmatrix}$$

We choose solutions $e^t \begin{bmatrix} \cos t + \sin t \\ \sin t \\ -\cos t \end{bmatrix}$ and $e^t \begin{bmatrix} \cos t - \sin t \\ \cos t \\ \sin t \end{bmatrix}$ that are $-\operatorname{Im}(x^*(t))$ and $-\operatorname{Re}(x^*(t)$ as two linearly independent solutions in addition to the solution $e^{-t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ corresponding to $\lambda_1 = -1$. The general solution is their linear combination as in the answer, because they are linearly independent and the dimension of the solutions space is 3 for the system of three linear ODEs.

Phase protrait in $\mathbb{R}^3$:



Code in Matlab for this picture:

```
t0 = 0; % starttime
L=10;
A=[ 3,-3, 1;
3,-2, 2;
-1, 2, 0 ];
[V,D]=eig(A)
tend = 20; % finish time
xlabel('x');
ylabel('y');
zlabel('z');
axis equal
axis([-L,L, -L, L, -L,L]);
hold on;
V=L*V;
plot3([V(1,1);-V(1,1)],[V(2,1);-V(2,1)],[V(3,1);-V(3,1)],'-r');
%plot for the real eigenvector
options = odeset('RelTol',1e-5);
for i=1:100;
[~, y] = ode45(@(t,y)A*y, [t0 tend], 5*[1-2*rand ;1-2*rand;1-2*rand], options);
```

```
plot3(y(:,1),y(:,2), y(:,3), 'b');
end
```

862. Answer. $r = C_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + C_3 e^t \begin{bmatrix} -1 \\ -t-1 \\ t \end{bmatrix}$

Solution. $x'(t) = Ax(t)$. Find general solution to this ODE.

$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, characteristic polynomial: $\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{bmatrix} =$
$\lambda^3 - 2\lambda^2 + \lambda = 0$.

Observe that $\lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda - 1)^2 = 0$.

Eigenvector with simple eigenvalue $\lambda_1 = 0$;

$(A - 0I)\, v_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; Gauss elimination $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \Longrightarrow$
$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

one free variable and one linearly independent eigenvector $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \leftrightarrow \lambda_2 = 1$,

where λ_2 is a multiple eigenvalue with albebraic multiplicity $n_2 = 2$.

$A - \lambda_2 I = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Longrightarrow$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ one free variable and one linearly independent eigenvector $v_2 =$
$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

$\dim(E(\lambda_2)) = m(\lambda_2) = 2$

generalized eigenvector $v_2^{(1)} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfies the equation $(A - \lambda_2 I)\, v_2^{(1)} = v_2$
because it would imply $(A - \lambda_2 I)^2\, v_2^{(1)} = (A - \lambda_2 I)\, v_2 = 0$, namely that $v_2^{(1)}$ is a generalized eigenvector.

in matrix form: $\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Corresponding equations are: $\begin{cases} -x+y+z= & 0 \\ x= & -1 \\ -x= & 1 \end{cases} \implies x=-1;\ y=-1;$

$z=0;\ v_2^{(1)} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

For arbitrary initial data $x_0 \in \mathbb{R}^3$, $x_0 = C_1 v_1 + C_2 v_2 + C_3 v_2^{(1)}$ the general solution is expressed as:

$$x(t) = e^{At} x_0 = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 + \left[I + (A - \lambda_2 I)\, t\right] e^{\lambda_2 t} v_2^{(1)}$$

Calculate the last term:

$\left[I + (A - \lambda_2 I)\, t\right] v_2^{(1)} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} =$

$\begin{bmatrix} -t+1 & t & t \\ t & 1 & 0 \\ -t & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -t-1 \\ t \end{bmatrix}$

Collect all terms and get the answer. Observe that one can multiply any term in the answer with -1 or with any other number, the answer will be still correct. One can get different answers choosing eigenvectors v_1 and v_2 in different ways.■

863. Answer. $r = C_1 e^{-t} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_3 e^{t} \begin{bmatrix} 2t \\ 2t \\ 2t+1 \end{bmatrix}$

864. Answer. $r = C_1 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} t+1 \\ t \\ 2t \end{bmatrix}$ complicated case with specific choice of eigenvectors.

Solution. $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 2 & -3 \end{bmatrix}$, characteristic polynomial: $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (1+\lambda)^3$, multiple eigenvalue $\lambda = -1$ with multiplicity 3.

The matrix has two linearly independent eigenvectors satisfying the homogeneous equation $(A - \lambda I)\, v = 0$.

$A - \lambda I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix}$, Gauss elimination leads to the equation $x_1 + x_2 - x_3 = 0$ that has two free variables x_2 and x_3

A possible choice of linearly independent eigenvectors is $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and

$v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ if we like to get an answer similar to one given above.

The column space $Col\,(A-\lambda I)$ is one-dimensional and consists of vectors $C\begin{bmatrix}1\\1\\2\end{bmatrix}=Cv$ with arbitrary real C. Therefore the system $(A-\lambda I)\,u=b$ is solvable if and only if $b=Cv$.

It means that we cannot build a generalized eigenvector solving equations $(A-\lambda I)\,v_1^{(1)}=v_1$ or $(A-\lambda I)\,v_2^{(1)}=v_2$ because by chance these two eigenvectors both do not bellong to $Col\,(A-\lambda I)$.

One can proceed by two ways. Observe that the vector $v=\begin{bmatrix}1\\1\\2\end{bmatrix}$ belongs to $Col\,(A-\lambda I)$ and is an eigenvector: $(A-\lambda I)\,v=0$.

Therefore the equation $(A-\lambda I)\,v^{(1)}=v$ has a solution. Consider corresponding extended matrix and carry out Gauss elimination on it:
$\begin{bmatrix}1&1&-1&1\\1&1&-1&1\\2&2&-2&2\end{bmatrix}\longrightarrow\begin{bmatrix}1&1&-1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$. There are two free variables and a 2-dimensional space of solutions $v^{(1)}$ with the simplest ones being $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\-1\end{bmatrix}$.

The choice $v^{(1)}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ leads to the general solution in the form

$$\begin{aligned}r(t)&=\exp(At)(C_1v_1+C_2v+C_3v^{(1)})\\&=C_1e^{-t}v_1+C_2e^{-t}v+C_3e^{-t}(v^{(1)}+tv)\end{aligned}$$

equivalent to the one given in the answer.

Another and possibly simpler solution in this situation could be just using the definition of generalized eigenvectors and trying to solve the equation

$$(A-\lambda I)^3\,v^{(1)}=0.$$

On this way we observe that $(A-\lambda I)^2=0$.

$$(A-\lambda I)^2=\begin{bmatrix}1&1&-1\\1&1&-1\\2&2&-2\end{bmatrix}\begin{bmatrix}1&1&-1\\1&1&-1\\2&2&-2\end{bmatrix}=0$$

This relation is non-trivial, because in general only $(A-\lambda I)^3=0$ must be valid for a matrix with characteristic polynomial $p(z)=(z+1)^3$.

It means that ALL vectors in $\mathbb{R}^3$ are generalized eigenvectors. It is a natural conclusion because we have only one eigenvalue of multiplicity 3, the same as the dimension of the problem.

To complement eigenvectors v_1 and v_2 with a linearly independent generalized eigenvector we could choose ANY vector in $\mathbb{R}^3$ linearly independent of eigenvectors v_1 and v_2 chosen before.

The vector $v^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a generalized eigenvector and is linearly independent of the eigenvectors $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ chosen before. With such choice of the basis we arrive to the same answer as before.

We could also choose the second eigenvector $v = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ instead of the vector v_2 to build a basis. The solution would have the following form:

$$\begin{aligned} r(t) &= \exp(At)(C_1 v_1 + C_2 v + C_3 v^{(1)}) = \\ &= C_1 e^{-t} v_1 + C_2 e^{-t} v + C_3 e^{-t}(v^{(1)} + tv) \\ &= C_1 e^{-t} v_1 + (C_2 + tC_3)\, e^{-t} v + C_3 e^{-t} v^{(1)} \end{aligned}$$

or with explicit coordinates:

$$r = C_1 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (C_2 + tC_3)\, e^{-t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Point out that this solution has different form comparing with the one in the answer. One can supply infinitely many correct answers by different choices of the basis representing initial conditions.■

865. Answer. $r = C_1 e^{2t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_3 e^{2t} \begin{bmatrix} 2t+1 \\ t \\ 3t \end{bmatrix}$