FOURIER ANALYSIS & METHODS LECTURE NOTES

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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According to Gerry, Fourier Analysis is "A collection of related techniques for solving the most important partial differential equations of physics (and chemistry)." For example, we're going to be solving partial differential equations, abbreviated PDEs

- Δ Laplace equations (related to computing energy of quantum particles)
- $\Box$ wave equations (describes the propagation of waves, hence also of light and electromagnetic waves)
- Ξ heat equation (describes the propagation of heat, is the quintessential diffusion equation)

What is a PDE?

Definition 1. A PDE is an equation for an unknown function (unsub) which depends on $n > 1$ independent real variables. Writing u for the unknown function,

$$u : \mathbb{R}^n \to \mathbb{C}.$$

The PDE for u is an equation that u is supposed to satisfy and contains u together with one or more partial derivatives of u. The PDE may also contain other, *specified* functions.

Example 1. *The Laplace equation for a function on* $\mathbb{R}^2$ *is:*

$$u_{xx} + u_{yy} = 0.$$

The Laplace operator on $\mathbb{R}^2$ *is:*

$$\Delta = \partial_{xx} + \partial_{yy},$$

so writing it this way the Laplace equation looks like

$$\Delta u = 0.$$

The wave equation for a function on $\mathbb{R}^3 \times [0, \infty)_t$ *is*

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}.$$

Sometimes there is a constant on one side or the other, but mathematicians often use interesting time units to be able to assume 'without loss of generality' this constant is 1. The heat equation for a function on $\mathbb{R} \times [0, \infty)_t$ *is*

$$u_t = u_{xx}.$$

Similarly, I like to assume the constant is 1.

1.1. **The sound check analogy.** Have you ever noticed that at a metal concert, even if the band has played thousands of concerts, even in the exact same venue, they *always* do a sound check? Do you know why? It's because the sound produced by the band obeys the wave equation. This equation is really hard to solve. Moreover, it is really sensitive to the geometry of the space where the band plays. Even if it's the same venue, the number of people inside is not the same, and these people are part of the geometry of the space. So, every time they play, the band has to do a sound check to see how the geometry of everything is affecting the solution of the wave equation which is basically how the band sounds.

The wave equation, and indeed all PDEs are HARD to solve. There is no single unifying theory to guide us to the solution of all PDEs. It's like the metal band: we have to do a sound check for each and every concert. There is no magic pre-set we can use for all our concerts. Similarly, we have to deal with each and every PDE individually and carefully. To solve them, we must study a variety of methods and learn how to use these methods and combine them when possible.

1.2. **The first method: Separation of variables (SV).** If you come to the (obligatory for Kf, option for TM and F) extra three lectures, you will learn how to classify every PDE on the planet. For the great majority of these, we have no hope to solve then analytically (that is, to write down a mathematical formula as the solution to the PDE).

In case you have forgotten, here is a reminder.

Definition 2. An ODE is an equation for an unknown function (unsub) which depends on *one* independent real variable. Writing u for the unknown function, an ODE for u is an equation that u is supposed to satisfy and contains u together with one or more derivatives of u. The ODE may also contain other, *specified* functions.

Question 3. *What is the difference between an ODE and a PDE?*[1]

So, to introduce the technique of separation of variables, let's think about a really down-to-earth example. A vibrating string, like the guitar or bass strings in our metal band. The ends of the string are held fixed, so they're not moving. You know this if you play or watch people play guitar. Let's mathematicize the string, by identifying it with the interval $[0, \ell] \subset \mathbb{R}$. The string length is ℓ. Let's define

$$u(x, t) := \text{ the height of the string at the point } x \in [0, \ell] \text{ at time } t \in [0, \infty[.$$

[1]Answer: the unknown function (unsub) in an ODE depends on only one variable, so the derivatives in the equation are 'ordinary derivatives.' The unknown function in a PDE depends on at least two variables, so we can no longer speak of ordinary derivatives, because the only derivatives that make sense when a function depends on two or more variables are *partial derivatives.* So, it's just a matter of how many variables does the unknown function in the equation depend on?

Then, let's just define the sitting-still height to be height 0. So, the fact that ends are sitting still means that

$$u(0,t) = u(\ell,t) = 0 \quad \forall t.$$

A positive height means above the sitting-still height, whereas a negative height means under the sitting-still height. The wave equation (I'm not going to derive it, but maybe you clever physics students can do that?) says that:

$$u_{xx} = \mathfrak{c}^2 u_{tt}.$$

The constant $\mathfrak{c}$ depends on how fast the string vibrates.

Question 4. *Is this equation a PDE or an ODE?*[2]

Technique 0 = Separation of Variables starts like this: we *assume* that

$$u(x,t) = X(x)T(t),$$

that is a product of two functions, each of which depends only on *one* variable. Why can we do this? Who knows, maybe it is rubbish! Maybe u is not of this form. Kind of like the sound check: we guess at the sound levels and then play a bit to see if it sounds good. Same here. We just have to try.

Assuming that u is of this form, we put this into the PDE:

$$u_{xx} = \mathfrak{c}^2 u_{tt} \iff X''(x)T(t) = \mathfrak{c}^2 X(x)T''(t).$$

Now, we would like to *separate variables* by getting everything dependent on x to one side of the equation and everything dependent on t to the other side. To achieve this, we divide both sides by $X(x)T(t)$:

$$\frac{X''}{X}(x) = \mathfrak{c}^2 \frac{T''}{T}(t).$$

Stop. Think. The left side depends only on x, whereas the right side depends only on t.

Exercise 1. *Explain in your own words why if one side of an equation depends on x and the other side depends on t, then both sides must be constant.*

What should we solve for first? X or T? We've got more information on X than we do on T, because we know that the ends are still. This means that

$$X(0) = X(\ell) = 0.$$

So, the equation for just f is

$$\frac{X''}{X}(x) = \text{ constant },$$

$$X(0) = X(\ell) = 0.$$

Let's give the constant a name. Call it λ. Then write

$$X''(x) = \lambda X(x), \quad X(0) = X(\ell) = 0.$$

Well, we can solve this. There are three cases to consider:

[2]Answer: it's a PDE because the function depends on two independent variables: position on the string x and time t.

$\lambda = 0$ This means $X''(x) = 0$. Integrating both sides once gives $X'(x) =$ constant $= m$. Integrating a second time gives $X(x) = mx + b$. Requiring $X(0) = X(\ell) = 0$, well, the first makes $b = 0$, and the second makes $m = 0$. So, the solution is $X(x) \equiv 0$. The 0 solution. The waveless wave. Not too interesting.

$\lambda > 0$ The solution here will be of the form

$$X(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}.$$

Exercise 2. *Show that it is equivalent to write the solution as $A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x)$, for two constants A and B. Determine the relationship between A and B and a and b. Show that in order to guarantee that $X(0) = X(\ell) = 0$ you need $a = A = B = b = 0$. You should do this exercise, because it I strongly suspect you can do it. Think of it as a warm-up for Folland's exercises.*

Thus, with our teamwork, (me providing hints and you doing the actual work by solving the exercise) we have gotten the 0 solution again. The waveless wave. No fun there.

$\lambda < 0$ Finally, we have solution of the form

$$a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To make $X(0) = 0$, we need $a = 0$. Uh oh... are we going to get that stupid 0 solution again? Well, let's see what we need to make $X(\ell) = 0$. For that we just need

$$b\sin(\sqrt{|\lambda|}\ell) = 0.$$

That will be true if

$$|\lambda| = \frac{k^2\pi^2}{\ell^2}, \quad k \in \mathbb{Z}.$$

Super! We still don't know what b ought to be, but at least we've found all the possible X's, up to constant factors.

Just to clarify the fact that we've now found *all* solutions, we recall here a theorem from your multivariable calculus class.

th:omc

Theorem 5 (Second order ODEs)**.** *Consider the second order linear homogeneous ODE,*

$$au'' + bu' + cu = 0, \quad a \neq 0.$$

If $b = c = 0$, then a basis of solutions is given by

$$\{x, 1\},$$

so that all solutions are of the form

$$u(x) = Ax + B, \quad A, B \in \mathbb{R}.$$

If $c = 0$, then a basis of solutions is $\{e^{-b/ax}, 1\}$ so that all real solutions are given by

$$u(x) = Ae^{-bx/a} + B.$$

If $c \neq 0$, then a basis of solutions is one of the following:

(1) $\{e^{r_1 x}, e^{r_2 x}\}$ if $b^2 \neq 4ac$, where

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

(2) $\{e^{rx}, xe^{rx}\}$ *if* $b^2 = 4ac$, *with* $r = -\frac{b}{2a}$.

Exercise 3. *Our equation is*

$$X'' = \lambda X \iff X'' - \lambda X = 0.$$

So, in the language of the above theorem, $a = 1$, $b = 0$, *and* $c = \lambda$. *Use this to find all solutions which satisfy* $X(0) = X(\ell) = 0$.

The solutions we've found are, up to constant factors:

$$X_k(x) = \sin\left(\frac{k\pi x}{\ell}\right), \quad \lambda_k = -\frac{k^2\pi^2}{\ell^2}.$$

> Do not worry about the constant factors at this point in time. Save them for later.[3]

Now, let's find the friends of X, the time functions, T which depend only on time. These come in pairs, so that X_1 comes together with T_1. This is because the value of the constant λ_1, comes from X_1. However, we've also got X_2, and the value of the constant λ_2 is different. So, for each pair we have

$$\frac{X_k''}{X_k} = \lambda_k = -\frac{k^2\pi^2}{\ell^2} = \mathfrak{c}^2\frac{T_k''}{T_k}.$$

This is almost the same equation we had before. Here we have, re-arranging:

$$T_k'' = -\frac{k^2\pi^2}{\mathfrak{c}^2\ell^2}T_k.$$

Exercise 4. *Use Theorem* `th:omc` *5 to show that a basis of solutions is given by*

$$\left\{e^{\frac{ik\pi t}{\mathfrak{c}\ell}}, e^{-\frac{ik\pi t}{\mathfrak{c}\ell}}\right\}.$$

Show that it is equivalent to use

$$\left\{\cos\left(\frac{k\pi t}{\mathfrak{c}\ell}\right), \sin\left(\frac{k\pi t}{\mathfrak{c}\ell}\right)\right\}$$

as a basis. Hint: remember $e^{i\theta} = \cos\theta + i\sin\theta$ *for* $i = \sqrt{-1}$ *for any* $\theta \in \mathbb{R}$.

Let us pause to think about what this means. The physics students may recognize that the numbers

$$\{|\lambda_k|\}_{k\geq 1}$$

are the resonant frequencies of the string. Basically, they determine how it sounds. The number $|\lambda_1|$ is the fundamental tone of the string. The higher $|\lambda_k|$ for $k \geq 2$ are harmonics. It is interesting to note that they are all square-integer multiplies of λ_1. Here's a question: if you can "hear" the value of $|\lambda_1|$, then can you tell me how long the string is? Well, yes, cause

$$|\lambda_1| = \frac{1}{\ell^2}, \implies \ell = \frac{1}{\sqrt{|\lambda_1|}}.$$

So, you can hear the length of a string. A couple of famous unsolved math problems: can one hear the shape of a convex drum? Can one hear the shape of a smoothly bounded drum? We can talk about these problems if you're interested.

[3]The reason we should do this is because the less baggage we are carrying around, (i.e. the fewer symbols we got to write), the less likely we are to screw something up. So, we should remember the patience principle and be patient, wait to get the constants later.

So, now that we've got all these solutions, what should we do with them? Good question...

1.3. **Superposition principle and linearity.** Superposition basically means adding up a bunch of solutions. You can think of it like adding up a bunch of solutions to get a super solution!

Definition 6. A second order linear PDE for an unknown function u of n variables is an equation for u and its mixed partial derivatives up to order two of the form

$$L(u) = f,$$

where f is a given function, and there are known functions $a(x)$, $b_i(x)$, $c_{ij}(x)$ for $x \in \mathbb{R}^n$ such that

$$L(u) = a(x)u(x) + \sum_{i=1}^{n} b_i(x)u_{x_i}(x) + \sum_{i,j=1}^{n} c_{ij}(x)u_{ij}(x).$$

In this context, L is called a *second order linear partial differential operator.*

The reason it's called linear is because it's well, linear.

Exercise 5. *For two functions u and v, which depend on n variables, show that*

$$L(u+v) = L(u) + L(v).$$

Moreover, for any constant $c \in \mathbb{R}$, show that

$$L(cu) = \mathfrak{c}L(u).$$

Definition 7. The wave operator, $\square$, defined for $u(x,y)$ with $(x,y) \in \mathbb{R}^2$ is

$$\square(u) = -u_{xx} + \mathfrak{c}^2 u_{tt}.$$

Exercise 6. *Verify that the wave operator is a second order linear partial differential operator.*

We have shown that the functions

$$u_k(x,t) = X_k(x)T_k(t)$$

satisfy

$$\square u_k = 0 \forall k.$$

Hence, if we add them up this remains true:

$$\square(u_1 + u_2 + u_3 + \ldots) = 0.$$

OBS![4]

Exercise 7. *Show that the equations*

$$X_k'' = \lambda_k X_k \iff f_k'' - \lambda_k X_k = 0$$

do not *add up. In particular, show that just the first two of these equations do not add up,*

$$X_1'' + X_2'' - (\lambda_1 + \lambda_2)(X_1 + X_2) \neq 0.$$

[4] I love this Swedish expression. Nothing quite like it in the languages I know. Well, the closest is maybe 小心 which is also very cute.

The reason these equations do not add up is because it's not the same L. *The equation for X_k is*

$$X_k'' - \lambda_k X_k = 0.$$

This depends on k, and since each $\lambda_1 \neq \lambda_2 \neq \lambda_3, \ldots$, the differential operator is

$$L_k = \frac{d^2}{dx^2} + \lambda_k.$$

This exercise shows that one must take care when smashing solutions (i.e. superposing) together!

When we look at the different $u_k(x,t)$ in the *wave equation,* it's all good, because it's always the same wave operator. Hence, we may indeed smash all our solutions together, include the (to be determined) coefficients, and write

$$u(x,t) = \sum_{k\geq 1} u_k(x,t) = \sum_{k\geq 1} \sin\left(\frac{k\pi x}{\ell}\right)\left(a_k \cos\left(\frac{k\pi t}{\mathfrak{c}\ell}\right) + b_k \sin\left(\frac{k\pi t}{\mathfrak{c}\ell}\right)\right),$$

and it satisfies

$$\Box u(x,t) = 0, \quad u(0,t) = u(\ell,t) = 0.$$

We've still got some unanswered questions:

(1) What are the constants a_k and b_k?
(2) If we can figure out what the constants are, then we are still left with this thing:
$$\sum_{k\geq 1} \sin\left(\frac{k\pi x}{\ell}\right)\left(a_k \cos(k\pi t/\ell) + b_k \sin(k\pi t/\ell)\right).$$
Is this hot mess going to converge?

2. Exercises to be done by oneself

1.1.1 Show that $u(x,t) = t^{-1/2}e^{-x^2/(4kt)}$ satisfies the heat equation
$$u_t = ku_{xx}.$$

1.2.5(a) Show that for $n = 1,2,3,\ldots$ $u_n(x,y) = \sin(n\pi x)\sinh(n\pi y)$ satisfies
$$u_{xx} + u_{yy} = 0, \quad u(0,y) = u(1,y) = u(x,0) = 0.$$

1.3.5 By separation of variables, derive the solutions $u_n(x,y) = \sin(n\pi x)\sinh(n\pi y)$ of
$$u_{xx} + u_{yy} = 0, \quad u(0,y) = u(1,y) = u(x,0) = 0.$$

1.3.7 Use separation of variables to find an infinite family of independent solutions to
$$u_t = ku_{xx}, \quad u(0,t) = 0, \quad u_x(\ell,t) = 0,$$
representing heat flow in a rod with one end held at temperature zero and the other end insulated.