MATEMATIK
Chalmers Tekniska Hogskola

Dugga

Fourieranalys/Fourier Metoder 2020 Solutions

Namn och personnummer:

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1. (1P) How many linearly independent solutions are there which satisfy:

$$u_t = ku_{xx}, \quad t > 0, \quad x \in (0, \ell), \quad u(0,t) = 0, \quad u_x(\ell, t) = 0?$$

   (a) there are no solutions
   (b) there is precisely one solution
   (c) there are infinitely many solutions
   (d) there are precisely 42 solutions

   There are infinitely many solutions. One of the first homework exercises was about just this fact.

2. (1P) Consider the following problem:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & 0 < t,\, 0 < x < 1 \\ u(x,0) = \sin(\pi x) & x \in [0,1] \\ u_t(x,0) = 0 & x \in [0,1] \\ u(0,t) = 0 = u(1,t) & t > 0 \end{cases}$$

   How can we solve this problem?

   (a) take the Fourier transform of the PDE in the x variable
   (b) solve a regular Sturm-Liouville problem and take the Laplace transform in the t variable
   (c) use separation of variables, superposition and a Fourier series
   (d) none of these will work

   Separation of variables, superposition and a Fourier series will work beautifully.

3. (1P) The Fourier series of the function which is equal to e^x on the interval $[-\pi, \pi]$ is

$$\sum_{n \in \mathbb{Z}} \frac{(-1)^n \sinh(\pi^2)}{\pi(\pi - in)} e^{inx}.$$

   Compute

$$\sum_{n \in \mathbb{Z}} \frac{\sinh(\pi^2)}{\pi(\pi - in)}.$$

   (a) the series does not converge
   (b) the series converges to e^{π}
   (c) the series converges to $\frac{e^{\pi}+e^{-\pi}}{2}$
   (d) none of these are correct

   Uhhhhhh, there was a small mistake here. The Fourier coefficients of e^x are

$$c_0 = \frac{\sinh \pi}{\pi} \quad c_n = \frac{(-1)^n \sinh(\pi)}{\pi(1 - in)}.$$

The series above is the series for $e^{\pi x}$. This is my mistake. Consequently two answers will be considered correct here. If one just presumed the Fourier series was indeed that for e^x, then the series would converge to

$$\frac{e^{\pi}+e^{-\pi}}{2}.$$

So answering c is considered correct. However, if you could not get over the fact that this is the Fourier series of $e^{\pi x}$, and the fact that the correct limit is $\frac{e^{\pi^2}+e^{-\pi^2}}{2}$, then you would have answered d. So, this will also be considered correct. However, even with this typo in the Fourier coefficients, one should still recognize that the series converges, and in either scenario (series for e^x or series for $e^{\pi x}$) the series does *not* converge to e^{π}. Thus, answers a and b are not correct with any reasonable interpretation of the problem.

4. (1P) Let

$$c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx.$$

Compute

$$\sum_{n\in\mathbb{Z}}|c_n|^2.$$

(a) the series does not converge
(b) the series converges to 2π
(c) the series converges to π^3
(d) none of these are correct

We have that e^{inx} form an orthogonal basis for $L^2(-\pi,\pi)$. Therefore Bessel's inequality is an equality. It says that

$$\sum_{n\in\mathbb{Z}}|\langle f,\phi_n\rangle|^2 = \int_{-\pi}^{\pi}|f(x)|^2dx,$$

for any $f\in L^2(-\pi,\pi)$, where ϕ_n are the elements of an orthonormal basis. Thus for

$$\phi_n = \frac{e^{inx}}{\sqrt{2\pi}}, \quad f(x)=x,$$

we have

$$\sum_{n\in\mathbb{Z}}|\langle f,\phi_n\rangle|^2 = \int_{-\pi}^{\pi}x^2dx = 2\int_0^{\pi}x^2dx = \frac{2\pi^3}{3}.$$

On the other hand

$$|\langle f,\phi_n\rangle|^2 = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}xe^{-inx}dx\right|^2 = 2\pi|c_n|^2.$$

So,

$$\sum_{n\in\mathbb{Z}}|\langle f,\phi_n\rangle|^2 = \sum_{n\in\mathbb{Z}}2\pi|c_n|^2 = \frac{2\pi^3}{3},$$

and therefore re-arranging we obtain

$$\sum_{n\in\mathbb{Z}}|c_n|^2 = \frac{\pi^2}{3}.$$

So, yes this was tricky, and yes, the answer is d.

5. (1P) Let

$$c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(x)e^{-inx}.$$

Compute:

$$\lim_{n\to\infty}c_n n^{42}.$$

(a) the limit does not exist
(b) the limit exists but there is insufficient information to compute its value
(c) the limit exists and is equal to 1
(d) none of these are correct

The Fourier coefficients $c_n = 0$ for all $n\notin\{1,-1\}$. So the limit exists and is equal to zero. Hence, d is the correct answer.