April 6, 2020

Exercises in ODE and modeling MMG511/TMV162. Spring 2020.
Weeks 1,2 . Linear systems of ODE with constant coefficients.

It is recommended to solve problems marked "Homework" and "solve in the class". They cover most of typical cases.

Find general solutions to following ODEs and sketch phase portraits for systems in plane:

786. $r' = Ar$ with $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$,

789. $\begin{cases} x' = & x + y \\ y' = & -2x + 3y \end{cases}$

790. $\begin{cases} x' = & x - 3y \\ y' = & 3x + y \end{cases}$

791. $\begin{cases} x' + x + 5y = & 0 \\ y' - x - y = & 0 \end{cases}$

792. $\begin{cases} x' = & 2x + y \\ y' = & -x + 4y \end{cases}$ **- Homework**

852. $r' = Ar$ with $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$,

853. $r' = Ar$ with $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$, **give as an exercise in the class ?**

854. $r' = Ar$ with $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$, **- demonstration in the class. Complex eigenvalues.**

856. $r' = Ar$ with $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & -2 \\ 1 & 5 & -3 \end{bmatrix}$, **- Homework**

857. $r' = Ar$ with $A = \begin{bmatrix} -1 & -2 & 2 \\ -2 & -1 & 2 \\ -3 & -2 & 3 \end{bmatrix}$,

858. $r' = Ar$ with $A = \begin{bmatrix} -3 & 2 & 2 \\ -3 & -1 & 1 \\ -1 & 2 & 0 \end{bmatrix}$, **- Homework complex eigenvalies**

859. $r' = Ar$ with $A = \begin{bmatrix} 3 & -3 & 1 \\ 3 & -2 & 2 \\ -1 & 2 & 0 \end{bmatrix}$, **- solve in the class complex eigenvalues**

861. $r' = Ar$ with $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

862. $r' = Ar$ with $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, **- solve in the class**

863. $r' = Ar$ with $A = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & 0 & 3 \end{bmatrix}$, **- Homework**

864. $r' = Ar$ with $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 2 & -3 \end{bmatrix}$, **complicated case when eigenvectors must be chosen in a clever way**

865. $r' = Ar$ with $A = \begin{bmatrix} 4 & 2 & -2 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix}$, **- solve in the class**

Calculate Jordans canonical matrices and find canonical basis for the following matrices.

It is nice to take one simple example, and a couple of larger examples. Some are suggested below.

6.4.23. $A = \begin{bmatrix} 11 & 4 \\ -4 & 3 \end{bmatrix}$; Homework

6.4.51. $A = \begin{bmatrix} 4 & 1 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 4 \end{bmatrix}$

6.4.63. $A = \begin{bmatrix} -2 & -1 & 1 \\ 5 & -1 & 4 \\ 5 & 1 & 2 \end{bmatrix}$; Homework

6.4.64. $A = \begin{bmatrix} 3 & -1 & 1 \\ -2 & 4 & -2 \\ -2 & 2 & 0 \end{bmatrix}$

6.4.65. $A = \begin{bmatrix} -4 & 4 & 2 \\ -1 & 1 & 1 \\ -5 & 4 & 3 \end{bmatrix}$

6.4.66. $A = \begin{bmatrix} 3 & 0 & -1 \\ -2 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$

861. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ Homework

862. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

863. $A = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & 0 & 3 \end{bmatrix}$

864. $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 2 & -3 \end{bmatrix}$,Homework

865. $A = \begin{bmatrix} 4 & 2 & -2 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix}$

Calculate e^A for following matrices A. Solve a couple of example 2x2 and two examples 3x3.

868. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; 869. $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$; 870. $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$; 871. $A = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$;

872. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$; 873. $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$; 859. $\begin{bmatrix} 3 & -3 & 1 \\ 3 & -2 & 2 \\ -1 & 2 & 0 \end{bmatrix}$ (difficult case with two complex conjugate eigenvalues)

Answers and solutions.

Theoretical background. We use the formula

$$x(t)=e^{At}x_0=\sum_{j=1}^{s}\left(\left[\sum_{k=0}^{m_j-1}(A-\lambda_j I)^k\,\frac{t^k}{k!}\right]x^{0,j}e^{\lambda_j t}\right)$$

for solutions with initial data

$$x(0)=x_0=\sum_{j=1}^{s}x^{0,j}$$

with $x^{0,j}\in E(\lambda_j,A)$ - components of x_0 in the generalized eigenspaces $E(\lambda_j,A)$ $=\ker(A-\lambda_j)^{m_j}$ of the matrix A. Here s is the number of distinct eigenvalues λ_j to A and m_j is the algebraic multiplicity of the eigenvalue λ_j. We point out that $\mathbb{C}^n=E(\lambda_1,A)\oplus\ E(\lambda_2,A)\oplus\ldots\oplus E(\lambda_s,A)$.

General solution can be expressed more explicitely by finding a basis of $\mathbb{C}^n$ in terms of eigenvectors v_j and generalized eigenvectors $v_j^{(k)}$ $k=1,\ldots l\le m_j-1$ corresponding to all distinct eigenvalues to A: λ_j, $j=1,\ldots s$, so that components $x^{0,j}$ of x_0 on to the generalized eigenspaces are expressed in the form

$$x^{0,j}=\ldots C_p v_j+C_{p+1}v_j^{(1)}+C_{p+2}v_j^{(2)}\ldots$$

including all linearly independent eigenvectors corresponding to λ_j (it might be several eigenvectors v_j corresponding to one λ_j) and corresponding linearly independent generalized eigenvectors for example calculated as it is suggested below.

Eigenvectors and generalized eigenvectors is convenient to calculate as a chain of vectors satisfying the following recursive chain of equations

$$\begin{aligned}(A-\lambda_j I)\,v_j &= 0,\\ (A-\lambda_j I)\,v_j^{0,1} &= v_j\\ (A-\lambda_j I)\,v_j^{0,2} &= v_j^{0,1}\\ & \quad e.t.c.\\ (A-\lambda_j I)\,v_j^{0,n_j-1} &= v_j^{0,n_j-2}\end{aligned}$$

It is not always possible to run this algorithm from the top downward, depending on the matrix and the choice of the eigenvectors. Sometimes the only way is to find a generalised eigenvector v_j^{0,n_j-1} using the definition solving the equation: $(A-\lambda_j I)^{n_j}\,v_j^{0,n_j-1}=0$ for n_j such that $(A-\lambda_j I)^{n_j-1}\,v_j^{0,n_j-1}\neq 0$. After that one can apply the same algorithm in the upward direction. Substituting this expression for x_0 in to the general formula above and carrying out all matrix-matrix and matrix-vector, multiplications one gets a general solution. Keep in mind that $(A-\lambda_j I)\,v_j=0$ and $(A-\lambda_j I)^2\,v_j^{0,1}=0$ e.t.c., so many terms in the expression
$\left[\sum_{k=0}^{m_j-1}(A-\lambda_j I)^k\,\frac{t^k}{k!}\right]x^{0,j}$ for $x^{0,j}=C_p v_j+C_{p+1}v_j^{(1)}+C_{p+2}v_j^{(2)}+\ldots$ are zero.

786. Answer. $r = C_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

789. Answer. $x = e^{2t}\left(C_1 \cos t + C_2 \sin t\right)$; $y = e^{2t}\left[\left(C_1 + C_2\right)\cos t + \left(C_2 - C_1\right)\sin t\right]$

790. Answer. $x = e^t\left(C_1 \cos 3t + C_2 \sin 3t\right)$; $y = e^t\left[C_1 \sin 3t - C_2 \cos 3t\right]$

791. Answer. $x = \left(2C_2 - C_1\right)\cos 2t - \left(2C_1 + C_2\right)\sin 2t$; $y = C_1 \cos 2t + C_2 \sin 2t$

792. Answer. $x = \left(C_1 + C_2 t\right)e^{3t}$; $y = \left(C_1 + C_2 + C_2 t\right)e^{3t}$

852. Answer. $r = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

853. Solution: $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}.A,$

characteristic polynomial: $\lambda^2 + 2\lambda + 1 = 0$ has a double eigenvalue: $\lambda = -1$, and one eigenvector: $v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Generalized eigenvector $v^{(1)} = \begin{bmatrix} x \\ y \end{bmatrix}$ satisfies the equation

$\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Longrightarrow 2x - 2y = 2;\ y = 1,\ x = 2;\ v^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Observe that v and $v^{(1)}$ are linearly independent (not parallel).

Therefore any initial data r_0 can be represented as $r_0 = C_1 v + C_2 v^{(1)}$ and solution to I.V.P. with initial data r_0 will be

$$\begin{aligned} r(t) &= e^{At} r_0 = C_1 e^{\lambda t} v + \left[I + (A - \lambda I)\, t\right] e^{\lambda t} C_2 v^{(1)} \\ &= C_1 e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{-t} C_2 \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \right) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \\ &= C_1 e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + C_2 \left(e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right) = C_1 e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 2t + 2 \\ 2t + 1 \end{bmatrix} \end{aligned}$$

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854. Answer. $r = C_1 e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$

Solution. $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$, characteristic polynomial: $\lambda^2 - 2\lambda + 5 = 0$;

eigenvectors:$v_1 = \left\{ \begin{bmatrix} 1 \\ 1 + i \end{bmatrix} \right\} \leftrightarrow \lambda_1 = 1 - 2i$,and $v_2 = \left\{ \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} \right\} \leftrightarrow \lambda_2 = 1 + 2i$.

A complex solution is $x^*(t) = e^{(1-2i)t} \begin{bmatrix} 1 \\ 1 + i \end{bmatrix}$.

Two linearly independent solutions can be chosen as real and imaginary part of $x^*(t)$ and can be used for representing a general solution as $x(t) = C_1 \operatorname{Re}\left[x^*(t)\right] + C_2 \operatorname{Im}\left[x^*(t)\right]$.

$$e^{(1-2i)t} \begin{bmatrix} 1 \\ 1 + i \end{bmatrix} = e^t\left(\cos 2t - i \sin 2t\right) \begin{bmatrix} 1 \\ 1 + i \end{bmatrix} = e^t \begin{bmatrix} \cos 2t - i \sin 2t \\ (1 + i)\cos 2t + (1 - i)\sin 2t \end{bmatrix}$$

$=$

$$e^t \begin{bmatrix} \cos 2t - i\sin 2t \\ \cos 2t + \sin 2t + i\,(\cos 2t - \sin 2t) \end{bmatrix} = e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} - i\,e^t \begin{bmatrix} \sin 2t \\ (\sin 2t - \cos 2t) \end{bmatrix}$$

Answer follows as linear combination of real and imaginary parts:
$x(t) = C_1 \operatorname{Re}[x^*(t)] + C_2 \operatorname{Im}[x^*(t)]$. ■

856. Answer. $r = C_1 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

857. Answer. $r = C_1 e^t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 2\cos t \\ 2\cos t \\ 3\cos t - \sin t \end{bmatrix} + C_3 \begin{bmatrix} 2\sin t \\ 2\sin t \\ 3\sin t + \cos t \end{bmatrix}$

Hints to finding complex eigenvectors.

858. Answer $r = C_1 e^{-2t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} \cos 2t \\ -\sin 2t \\ \cos 2t \end{bmatrix} + C_3 e^{-t}\begin{bmatrix} \sin 2t \\ \cos 2t \\ \sin 2t \end{bmatrix}$

Two linearly independent solutions can be chosen as above, as real and imaginary part of one of the complex conjugate complex solutions $x^*(t)$ corresponding to a complex eigevalue and can be used for representing a general solution. A complication in the present case is to find complex eigenvectors satisfying a homogeneous system of three equations.

$\begin{bmatrix} -3 & 2 & 2 \\ -3 & -1 & 1 \\ -1 & 2 & 0 \end{bmatrix}$, characteristic polynomial: $p(\lambda) = \lambda^3 + 4\lambda^2 + 9\lambda + 10$,

roots: $\lambda_1 = -2$, $\lambda_2 = -1 - 2i$, $\lambda_3 = \overline{\lambda_2} = -1 + 2i$. The real root one can guess, two other are found from a quadratic equation.

An eigenvector corresponding to the eigenvalue $\lambda_2 = -1 - 2i$ satisfies homogeneous system with matrix $A - \lambda_2 I$:

$$A - \lambda_2 I = \begin{bmatrix} -3-(-1-2i) & 2 & 2 \\ -3 & -1-(-1-2i) & 1 \\ -1 & 2 & -(-1-2i) \end{bmatrix} = \begin{bmatrix} -2+2i & 2 & 2 \\ -3 & 2i & 1 \\ -1 & 2 & 1+2i \end{bmatrix}$$

Change order of rows and multiply the first row by -1:

$\begin{bmatrix} 1 & -2 & -1-2i \\ 1-i & -1 & -1 \\ 3 & -2i & -1 \end{bmatrix}$,

Multiply the second row by the conjugate $1+i$ **of it's first non-zero element** $1-i$ **to simplify Gauss elimination and use that** $(1+i)(1-i) = 1+1 = 2$.

In general for $z = a + ib$ and it's complex conjugate $\overline{z} = a - ib$

$$z\,\overline{z} = (a+ib)(a-ib) = a^2 + b^2 = |z|^2$$

$$\rightarrow \begin{bmatrix} 1 & -2 & -1-2i \\ 2 & -1-i & -1-i \\ 3 & -2i & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1-2i \\ 0 & 3-i & 1+3i \\ 0 & 6-2i & 2+6i \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -2 & -1-2i \\ 0 & 3-i & 1+3i \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$$

Multiply the second row by the conjugate $3+i$ **of it's first non-zero element** $3-i$ **an use that** $(3+i)(3-i) = 9+1 = 10$:

$$\rightarrow \begin{bmatrix} 1 & -2 & -1-2i \\ 0 & (3-i)(3+i) & (1+3i)(3+i) \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1-2i \\ 0 & 10 & 10i \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & -1-2i \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix}$$

Choosing components in v_2 as $x_3 = 1$ we get $x_2 = -i$, and $x_1 = 1$ and $v_2 = \begin{matrix} 1 \\ -i \\ 1 \end{matrix}$.

The second complex eigenvector corresponding to the conjugate eigenvalue λ_3 is complex conjugate to v_2 because the matrix A is real: $v_2 = \overline{v_3}$ and $\lambda_2 = \overline{\lambda_3}$ are congugate.

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859. Answer. $r = C_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} \cos t - \sin t \\ \cos t \\ \sin t \end{bmatrix} + C_3 e^{t} \begin{bmatrix} \cos t + \sin t \\ \sin t \\ -\cos t \end{bmatrix}$

Solution. $A = \begin{bmatrix} 3 & -3 & 1 \\ 3 & -2 & 2 \\ -1 & 2 & 0 \end{bmatrix}$., eigenvectors:

The characteristic polynomial is : $\lambda^3 - \lambda^2 + 2 = (\lambda + 1)\left(\lambda^2 - 2\lambda + 2\right) = 0$.

Eigenvectors to the eigenvalue $\lambda_2 = 1 - i$ are found from the homogeneous system of equations with matrix

$$\begin{bmatrix} 2+i & -3 & 1 \\ 3 & -3+i & 2 \\ -1 & 2 & -1+i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1-i \\ 3 & -3+i & 2 \\ 2+i & -3 & 1 \end{bmatrix}$$

Hint to finding complex eigenvectors.

Multiply the last row by the conjugate of the first element to simplify Gauss elimination: $\rightarrow \begin{bmatrix} 1 & -2 & 1-i \\ 3 & -3+i & 2 \\ (2+i)(2-i) & -3(2-i) & (2-i) \end{bmatrix} = \rightarrow \begin{bmatrix} 1 & -2 & 1-i \\ 3 & -3+i & 2 \\ 5 & -6+3i & 2-i \end{bmatrix} \rightarrow$

$$\begin{bmatrix} 1 & -2 & 1-i \\ 0 & 3+i & -1+3i \\ 0 & 4+3i & -3+4i \end{bmatrix}$$

Multiply rows 2 and 3 by conjugates of pivot elements in each row to simplify Gauss elimination:

$$\rightarrow \begin{bmatrix} 1 & -2 & 1-i \\ 0 & (3+i)(3-i) & (-1+3i)(3-i) \\ 0 & (4+3i)(4-3i) & (-3+4i)(4-3i) \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1-i \\ 0 & 10 & 10i \\ 0 & 25 & 25i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1-i \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix}$$

Chose $x_3 = 1$, $x_2 = -i$, $x_1 = -1 - i$.

The second eigenvector corresponding to the conjugate eigenvalue is complex conjugate because the matrix A is real: $v_2 = \overline{v_3}$ and $\lambda_2 = \overline{\lambda_3}$ are congugate.

Eigenvectors and eigenvalues are: $v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \leftrightarrow \lambda_1 = -1, v_2 = \begin{bmatrix} -1-i \\ -i \\ 1 \end{bmatrix} \leftrightarrow$

$\lambda_2 = 1 - i$, $v_3 = \begin{bmatrix} -1+i \\ i \\ 1 \end{bmatrix} \leftrightarrow \lambda_3 = 1 + i$, Eigenvalues are all simple, therefore eigenvectors are linearly independent and general complex solutions are expressed as $x(t) = \sum_{k=1}^{3} C_k e^{\lambda_k t} v_k$. If we look for general real solutions that is natural for a real matrix A, we can use solution real and imaginary parts of the complex solution $x^*(t) = v_2 e^{\lambda_2 t}$ as two linearly independent real solutions to the ODE in addition to $e^{\lambda_1 t} v_1$.

$$x^*(t) = e^{(1-i)t} \begin{bmatrix} -1-i \\ -i \\ 1 \end{bmatrix} = e^t(\cos t - i \sin t) \begin{bmatrix} -1-i \\ -i \\ 1 \end{bmatrix} = e^t \begin{bmatrix} -(1+i)\cos t - (1-i)\sin t \\ -i\cos t - \sin t \\ \cos t - i \sin t \end{bmatrix}$$

$$= e^t \begin{bmatrix} -(1)\cos t - (1)\sin t - (i)\cos t - (-i)\sin t \\ -\sin t - i\cos t \\ \cos t - i\sin t \end{bmatrix} = e^t \begin{bmatrix} -\cos t - \sin t \\ -\sin t \\ \cos t \end{bmatrix} + i e^t \begin{bmatrix} -\cos t + \sin t \\ -\cos t \\ -\sin t \end{bmatrix}$$

We choose solutions $e^t \begin{bmatrix} \cos t + \sin t \\ \sin t \\ -\cos t \end{bmatrix}$ and $e^t \begin{bmatrix} \cos t - \sin t \\ \cos t \\ \sin t \end{bmatrix}$ that are $-\operatorname{Im}(x^*(t))$ and $-\operatorname{Re}(x^*(t)$ as two linearly independent solutions in addition to the solution $e^{-t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ corresponding to $\lambda_1 = -1$. The general solution is their linear combination as in the answer, because they are linearly independent and the dimension of the solutions space is 3 for the system of three linear ODEs.

861. Answer $r = C_1 e^{3t}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + C_3 e^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

862. Answer. $r = C_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + C_2 e^{t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + C_3 e^{t}\begin{bmatrix} -1 \\ -t-1 \\ t \end{bmatrix}$

Solution. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, characteristic polynomial: $\lambda^3 - 2\lambda^2 + \lambda = 0$.

Observe that $\lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda - 1)^2 = 0$

Eigenvectors: $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \leftrightarrow \lambda_1 = 0$ with simple eigenvalue λ_1; $v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \leftrightarrow \lambda_2 = 1$,

where λ_2 is a multiple eigenvalue with albebraic multiplicity $n_2 = 2$.

$A - \lambda_2 I = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. generalized eigenvector $v_2^{(1)} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfies the equation

$(A - \lambda_2 I)\, v_2^{(1)} = v_2$ or in matrix form: $\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Corresponding equations are: $\begin{cases} -x + y + z = & 0 \\ x = & -1 \\ -x = & 1 \end{cases} \Longrightarrow x = -1;\ y = -1;$ $z = 0;\ v_2^{(1)} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

For arbitrary initial data $x_0 \in \mathbb{R}^3$, $x_0 = C_1 v_1 + C_2 v_2 + C_3 v_2^{(1)}$ the general solution is expressed as:

$$x(t) = e^{At}x_0 = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 + \left[I + (A - \lambda_2 I)\, t\right] e^{\lambda_2 t} v_2^{(1)}$$

Calculate the last term:

$\left[I + (A - \lambda_2 I)\, t\right] v_2^{(1)} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} =$
$\begin{bmatrix} -t+1 & t & t \\ t & 1 & 0 \\ -t & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -t-1 \\ t \end{bmatrix}$

Collect all terms and get the answer. Observe that one can multiply any term in the answer with -1 or with any other number, the answer will be still correct. One can get different answers choosing eigenvectors v_1 and v_2 in different ways.■

863. Answer. $r = C_1 e^{-t} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_3 e^{t} \begin{bmatrix} 2t \\ 2t \\ 2t+1 \end{bmatrix}$

864. Answer. $r = C_1 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} t+1 \\ t \\ 2t \end{bmatrix}$

Solution. $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 2 & -3 \end{bmatrix}$, characteristic polynomial: $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (1+\lambda)^3$, multiple eigenvalue $\lambda = -1$ with multiplicity 3.

The matrix has two linearly independent eigenvectors satisfying the homogeneous equation $(A - \lambda I)\, v = 0$.

$A - \lambda I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix}$, Gauss elimination leads to the equation $x_1 + x_2 - x_3 = 0$ that has two free variables x_2 and x_3

A possible choice of linearly independent eigenvectors is $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ if we like to get an answer similar to one given above.

The column space $Col\,(A - \lambda I)$ is one-dimensional and consists of vectors $C \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = Cv$ with arbitrary real C. Therefore the system $(A - \lambda I)\, u = b$ is solvable if and only if $b = Cv$.

It means that we cannot build a generalized eigenvector solving equations $(A - \lambda I)\, v_1^{(1)} = v_1$ or $(A - \lambda I)\, v_2^{(1)} = v_2$ because by chance these two eigenvectors both do not bellong to $Col\,(A - \lambda I)$.

One can proceed by two ways. Observe that the vector $v = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ belongs to $Col\,(A - \lambda I)$ and is an eigenvector: $(A - \lambda I)\, v = 0$.

Therefore the equation $(A - \lambda I)\, v^{(1)} = v$ has a solution. Consider corresponding extended matrix and carry out Gauss elimination on it:

$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 2 & 2 & -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. There are two free variables and a 2-dimensional space of solutions $v^{(1)}$ with the simplest ones being $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$.

The choice $v^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ leads to the general solution in the form

$$\begin{aligned} r(t) &= \exp(At)(C_1 v_1 + C_2 v_1 + C_3 v^{(1)}) \\ &= C_1 e^{-t} v_1 + C_2 e^{-t} v_2 + C_3 e^{-t}(v^{(1)} + tv) \end{aligned}$$

equivalent to the one given in the answer.

Another and possibly simpler solution in this situation could be just using the definition of generalized eigenvectors and trying to solve the equation $(A - \lambda I)^2 v^{(1)} = 0$. On this way we observe that $(A - \lambda I)^2 = 0$. This relation is non-trivial, because in general only $(A - \lambda I)^3 = 0$ must be valid for a matrix with characteristic polynomial $p(z) = (z+1)^3$.

It means that ALL vectors in $\mathbb{R}^3$ are generalized eigenvectors. It is a natural conclusion because we have only one eigenvalue of multiplicity 3, the same as the dimension of the problem.

To complement eigenvectors v_1 and v_2 with a linearly independent generalized eigenvector we could choose ANY vector in $\mathbb{R}^3$ linearly independent of eigenvectors v_1 and v_2 chosen before.

The vector $v^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a generalized eigenvector and is linearly independent of the eigenvectors $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ chosen before. With such choice of the basis we arrive to the same answer as before.

We could also choose vector $v = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ instead of the vector v_2 to build a basis. The solution would have the following form:

$$\begin{aligned} r(t) &= \exp(At)(C_1 v_1 + C_2 v + C_3 v^{(1)}) = \\ &= C_1 e^{-t} v_1 + C_2 e^{-t} v + C_3 e^{-t}(v^{(1)} + tv) \\ &= C_1 e^{-t} v_1 + (C_2 + tC_3)\, e^{-t} v + C_3 e^{-t} v^{(1)} \end{aligned}$$

or with explicit coordinates:

$$r = C_1 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (C_2 + tC_3)\, e^{-t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Point out that this solution has different form comparing with the one in the answer. One can supply infinitely many correct answers by different choices of the basis representing initial conditions.■

865. Answer. $r = C_1 e^{2t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_3 e^{2t} \begin{bmatrix} 2t+1 \\ t \\ 3t \end{bmatrix}$

1 Calculation of matrix exponent exp(A). Answers and some solutions to exercises.

Answers.

868. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; $e^A = \begin{bmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{bmatrix}$; 869. $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$; $e^A = \begin{bmatrix} e^2 & e^2 \\ 0 & e^2 \end{bmatrix}$;

870. $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$; $e^A = \begin{bmatrix} 2e^2 - e & e - e^2 \\ 2e^2 - 2e & 2e - e^2 \end{bmatrix}$; 871. $A = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$; $e^A = \begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}$; 872. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$; $e^A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^2 \end{bmatrix}$; 873. $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$; $e^A = \begin{bmatrix} e^2 & e^2 & \frac{e^2}{2} \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix} = e^2 \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

859. $\begin{bmatrix} 3 & -3 & 1 \\ 3 & -2 & 2 \\ -1 & 2 & 0 \end{bmatrix}$;

$$\exp(A) = e \begin{bmatrix} (\cos 1 + \sin 1) - e^{-2} + (\cos 1 - \sin 1) & -(\cos 1 + \sin 1) + e^{-2} & -e^{-2} + (\cos 1 - \sin 1) \\ (\cos 1) + (\sin 1) - e^{-2} & -(\sin 1) + e^{-2} & (\cos 1) - e^{-2} \\ -(\cos 1) + (\sin 1) + e^{-2} & (\cos 1) - e^{-2} & (\sin 1) + e^{-2} \end{bmatrix}$$

Hints to the calculation of $e^A = T\exp(J)T^{-1}$.

One can apply formulas for solution of linear ODE first and use the fact that columns with index i in e^A are values of solutions $x(1)$ to $x' = Ax$ at time $t = 1$ corresponding to initial data $x(0) = e_i = [0, ...0, 1, +...0]^T$.Vectors e_i are colums with index i from the unit matrix I.

$$x(t) = \exp(At)\xi$$

Examples of calculations of exp(A)

Solutions to 869, 872, 873 are just explicit formulas for Jordan's blocks and matrices in canonical Jordan's form.

Jordan's block:

$$J = \begin{bmatrix} \lambda & 1 & 0 & ... & 0 & 0 \\ 0 & \lambda & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & \lambda & 1 \\ 0 & 0 & 0 & ... & 0 & \lambda \end{bmatrix}$$

$$\exp(Jt) = e^{\lambda t}\begin{bmatrix} 1 & t & t^2/2 & ... & \frac{t^{m-2}}{(m-2)!} & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & ... & \frac{t^{m-3}}{(m-3)!} & \frac{t^{m-2}}{(m-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & t & t^2/2 \\ 0 & 0 & 0 & ... & 1 & t \\ 0 & 0 & 0 & ... & 0 & 1 \end{bmatrix}$$

Matrices in canonical Jordan's form:

$$\mathbb{J} = \begin{bmatrix} J_1 & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & J_2 & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & J_k \end{bmatrix}$$

$$\exp(\mathbb{J}) = \begin{bmatrix} \exp(J_1) & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \exp(J_2) & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & ... & \vdots \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \exp(J_k) \end{bmatrix}$$

The next example of calculation of $\exp(A)$ is specific because the real matrix has complex eigenvalues!

We can diagonalise it and write the answer in complex form, but it will be difficult to see that the result is a real matrix.

Solution to 868. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; This matrix has complex eigenvalues $\lambda_{1,2} = \pm i$.

The set of matrices of the structure $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ have the same properties with respect to matrix multiplication and addition as complex numbers of the form $a + ib$. In particular matrices of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ behave as real numbers and matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ behave as imaginary unit i.

It makes that we can apply the Euler formula

$$\exp(a + ib) = \exp(a)(\cos(b) + i \sin(b))$$

for computing

$$\exp\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) = \exp\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) \exp\left(\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}\right) = \exp(a) \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}$$

It implies immediately that $\exp(A) = \exp\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{bmatrix}$

Another general way to calculate exponents of matrices. (particularly useful for matrices having complex eigenvalues)

We use here general solution to the equation $x' = Ax$.

We clarify first in which way it can be used.

- For any matrix B the product Be_k gives the column k in the matrix B.
- Therefore the column k in $\exp(A)$ is the product $\exp(A)e_k$, where vector e_k is a standard basis vector, or colum with index k from the unit matrix I.
- On the other hand $\exp(At)\xi$ is a solution to the equation $x' = Ax$ with initial condition $x(0) = \xi$
- The expressions $x_k(t) = \exp(At)e_k$ is a solution to the equation $x' = Ax$ with initial condition $x(0) = e_k$
- Therefore the value of the solution in time $t = 1$: $x_k(1) = \exp(A)e_k$ gives the column k in the matrix $\exp(A)$
- Having the general solution for example in the case of dimension 3:

$$x(t) = C_1\Psi_1(t) + C_2\Psi_2(t) + C_3\Psi_3(t)$$

in terms of linearly independent solutions $\Psi_1(t)$, $\Psi_2(t)$, $\Psi_3(t)$, we can for every k find sets of constants $C_{1,k}$,$C_{2,k}$,$C_{3,k}$, corresponding to each of the initial data e_k. Namely we solve equations

$$C_{1,k}\Psi_1(0) + C_{2,k}\Psi_2(0) + C_{3,k}\Psi_3(0) = e_k\ , \quad k = 1,2,3$$

- that are equivalent to the matrix equation

$$[\Psi_1(0), \Psi_2(0), \Psi_3(0)] \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = [e_1, e_2, e_3] = I$$

- Values at $t = 1$ of corresponding solutions:

$$x_k(1) = C_{1,k}\Psi_1(1) + C_{2,k}\Psi_2(1) + C_{3,k}\Psi_3(1) = \exp(1 \cdot A)e_k$$

give us columns $\exp(1 \cdot A)e_k$ in $\exp(A)$.

- In matrix form this result can be expressed as

$$\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = [\Psi_1(0), \Psi_2(0), \Psi_3(0)]^{-1}$$

$$\begin{aligned} \exp(A) &= [\Psi_1(1), \Psi_2(1), \Psi_3(1)] \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} \\ &= [\Psi_1(1), \Psi_2(1), \Psi_3(1)]\,[\Psi_1(0), \Psi_2(0), \Psi_3(0)]^{-1} \end{aligned}$$

We demonstrate this idea using the result on the general solution from the problem 859.

We can calculate $\exp\left(\begin{bmatrix} 3 & -3 & 1 \\ 3 & -2 & 2 \\ -1 & 2 & 0 \end{bmatrix}\right)$, eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 1 - i$, $\lambda_3 = 1 + i$

General solution to the system $x' = Ax$ is:

$$\begin{aligned} x(t) &= C_1\Psi_1(t) + C_2\Psi_2(t) + C_3\Psi_3(t) \\ &= C_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} \cos t - \sin t \\ \cos t \\ \sin t \end{bmatrix} + C_3 e^{t} \begin{bmatrix} \cos t + \sin t \\ \sin t \\ -\cos t \end{bmatrix} \end{aligned}$$

and introducing shorter notations for each term: $x(t) = C_1\Psi_1(t) + C_2\Psi_3(t) + C_3\Psi_3(t)$.

We calculate initial data for arbitrary solution by

$x(0) = C_1\Psi_1(0) + C_2\Psi_3(0) + C_3\Psi_3(0) = C_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$x(0) = [\Psi_1(0), \Psi_3(0), \Psi_3(0)] \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$

$\exp(A)$ has columns that are values of $x(1)$ for solutions that satisfy initial conditions $r(0) = e_1, e_2, e_3$ and therefore $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,1} \\ C_{2,1} \\ C_{3,1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1$; $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,2} \\ C_{2,2} \\ C_{3,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_2$; $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,3} \\ C_{2,3} \\ C_{3,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_3$;

We solve all three of these systems for $\begin{bmatrix} C_{1,k} \\ C_{2,k} \\ C_{3,k} \end{bmatrix}$ in one step as a matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = I$$

It is equivalent to the Gauss elimination of this extended matrix: $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$. The result will be the inverted matrix:

$$\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

It can also found by applying Cramer's rule.

We arrive to the expression of the matrix exponent by collecting these results through the matrix multiplication:

$\exp(At) = [\Psi_1(t), \Psi_2(t), \Psi_3(t)] \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix}$

$\exp(At) = \begin{bmatrix} e^{-t} & e^t(\cos t - \sin t) & e^t(\cos t + \sin t) \\ e^{-t} & e^t \cos t & e^t \sin t \\ -e^{-t} & e^t \sin t & -e^t \cos t \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} =$

$= \begin{bmatrix} e^t(\cos t + \sin t) - e^{-t} + e^t(\cos t - \sin t) & -e^t(\cos t + \sin t) + e^{-t} & -e^{-t} + e^t(\cos t - \sin t) \\ (\cos t)e^t + (\sin t)e^t - e^{-t} & -(\sin t)e^t + e^{-t} & (\cos t)e^t - e^{-t} \\ -(\cos t)e^t + (\sin t)e^t + e^{-t} & (\cos t)e^t - e^{-t} & (\sin t)e^t + e^{-t} \end{bmatrix}$

and finally for $t = 1$ we get $\exp(A)$

$$\exp(A) = e\begin{bmatrix} (\cos 1+\sin 1)-e^{-2}+(\cos 1-\sin 1) & -(\cos 1+\sin 1)+e^{-2} & -e^{-2}+(\cos 1-\sin 1) \\ (\cos 1)+(\sin 1)-e^{-2} & -(\sin 1)+e^{-2} & (\cos 1)-e^{-2} \\ -(\cos 1)+(\sin 1)+e^{-2} & (\cos 1)-e^{-2} & (\sin 1)+e^{-2} \end{bmatrix}$$

■

Solution to 870. $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$

Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2$; eigenvalues: $\lambda_1 = 1, \lambda_2 = 2$

$A - I = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$; $A - 2I = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$;

Eigenvectors: $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \leftrightarrow \lambda_1 = 1, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftrightarrow \lambda_2 = 2$

The matrix is diagonalisable: $A = TDT^{-1}; D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; T = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$;

(Cramer's rule)$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$;

$T^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = (-1)\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$;

$\exp(A) = T\exp(D)T^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\exp\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} =$

$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\left(\begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}\right)\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$;

$\left(\begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}\right)\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -e & e \\ 2e^2 & -e^2 \end{bmatrix}$

$T\exp(D)T^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -e & e \\ 2e^2 & -e^2 \end{bmatrix} = \begin{bmatrix} 2e^2 - e & e - e^2 \\ -2e + 2e^2 & 2e - e^2 \end{bmatrix}$

Solution to 871. $A = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$;

characteristic polynomial: $\lambda^2 = 0$, multiple eigenvalue $\lambda = 0$.

eigenvector $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, the only linearly independent because there is only on free variable.

A generalised eigenvector can be found from the equation $(A - 0I)\, v^{(1)} = v$.

$\begin{bmatrix} -2 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix}$, Gaussian elimination: $\begin{bmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

A generalised eigenvector can be chosen as $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $T = \left[v, v^{(1)}\right] = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$.

Jordan matrix is $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $\exp(J) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\exp(A) = TJT^{-1}$; $T^{-1} = \left(\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$(by Cramer's rule)

$\exp(A) = TJT^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} =$
$\begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}$

2 Jordan matrices. Answers and some solutions

Answers to problems 861-865 on canonical Jordan matrices can be derived from answers to solutions of corresponding differential equations above.

Answers.

6.4.23. $J = \begin{bmatrix} 7 & 1 \\ 0 & 7 \end{bmatrix}$; $V = \begin{bmatrix} 4 & 0 \\ -4 & 1 \end{bmatrix}$

Solution.

$A = \begin{bmatrix} 11 & 4 \\ -4 & 3 \end{bmatrix}$ characteristic polynomial: $p(X) = X^2 - 14X + 49 = (X-7)^2$

$A - 7I = \begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$; $\quad (A-7I)v^{(1)} = v$; $\quad \begin{bmatrix} 4 & 4 & 4 \\ -4 & -4 & -4 \end{bmatrix}$,

row echelon form: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\quad v^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

6.4.51. $J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$; $V = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

Solution.

$A = \begin{bmatrix} 4 & 1 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 4 \end{bmatrix}$, characteristic polynomial: $p(X) = X^3 - 9X^2 + 27X - 27 = (X-3)^3 = 0$

$A - 3I = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \in Col(A-3I)(!!!!!)$, $w = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$,

- eigenvectors

$(A-3I)\,v^{(1)} = v$; $\quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -2 & -2 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

$v^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$(A-3I)\,w^{(1)} = w$; $\quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & -2 & -2 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

no solution to this system.

Point out that in this exercise similarly to exercise 864, the matrix $(A - \lambda I)$ has one-dimensional column space.

Here in the exercise 6.4.51, this matrix is $(A - \lambda I) = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$.

Its columns space $Col(A)$ is the line through the origin parallel to the vector $v_c = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Point out also that this vector is an eigenvector.

For the inhomogeneous system for the generalized eigenvector $(A-\lambda I)v^{(1)} = v$ for some eigenvector v to have a solution the right hand side must be from the column space. It makes that possible choice of a chain of generalized eigenvectors is limited in this case by a one dimensional subspace of eigenvectors parallel to the vector v_c. Point out that in the exercise 6.4.51 we need a chain of generalised eigenvectors to find a transformation $T^{-1}AT = J$ of the matrix A to a canonical Jordan's form J.

In the case with the Exercise 864 we had more freedom because we looked for any basis of eigenvectors and generalised eigenvectors to build a general solution to the system $x' = Ax$.

In both examples $(A - \lambda I)^2 = 0$ (check it!). It implies that any vector $z \in \mathbb{R}^3$ satisfies the equation

$$(A - \lambda I)^2 z = 0$$

and is a generalised eigenvector in this case. We are free just to choose a vector that is not a usual eigenvector (not belonging to the envelope of two eigenvectors you have already found). It would be enough to derive a general solution to the system $x' = Ax$.

But as we pointed out before, if we like to find the transformation matrix in exercise 6.4.1, we need to find a chain of generalised eigenvectors.

A way around the corner is to put all the problem up set down. Choose first ANY vector $v^{(1)}$ that satisfies the equation

$$(A - \lambda I)^2 v^{(1)} = 0$$

and is NOT and eigenvector, as a generalized eigenvector. We can try vector $v^{(1)} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Then calculate corresponding eigenvector v in the chain as

$$(A - \lambda I)v^{(1)} = v$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$$

Point out that the eigenvector v that we have got is automatically in the one dimensional $Col(A)$ subspace. We have built one chain of generalised eigenvectors to A.

After that we need to find the second eigenvector that is linearly independent of v (in this simple case not parallel to). The eigenvector $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ has this property.

6.4.63. $J = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$; $V = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$;

Solution.

$A = \begin{bmatrix} -2 & -1 & 1 \\ 5 & -1 & 4 \\ 5 & 1 & 2 \end{bmatrix}$, characteristic polynomial: $X^3 + X^2 - 8X - 12 = (X-3)(X+2)^2 = 0$

$A - 3I = \begin{bmatrix} -5 & -1 & 1 \\ 5 & -4 & 4 \\ 5 & 1 & -1 \end{bmatrix}$, Gaussian elimination: $\begin{bmatrix} -5 & -1 & 1 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$,

$v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$;

$A + 2I = \begin{bmatrix} 0 & -1 & 1 \\ 5 & 1 & 4 \\ 5 & 1 & 4 \end{bmatrix}$, Gaussian elimination: $\begin{bmatrix} 5 & 1 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$w = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$; $(A+2I)\, w^{(1)} = w$; $\begin{bmatrix} 0 & -1 & 1 & -2 \\ 5 & 1 & 4 & 2 \\ 5 & 1 & 4 & 2 \end{bmatrix}$, Gaussian elimination: $\begin{bmatrix} 5 & 1 & 4 & 2 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $w^{(1)} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$;

6.4.64. $J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$; $V = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

$A = \begin{bmatrix} 3 & -1 & 1 \\ -2 & 4 & -2 \\ -2 & 2 & 0 \end{bmatrix}$

6.4.65. $J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$; $V = \begin{bmatrix} -4 & 2 & 1 \\ -3 & 2 & 0 \\ -4 & -1 & 1 \end{bmatrix}$