TMA947 / MMG621 — Nonlinear optimisation

Exercise 3 – KKT conditions, Lagrangian duality

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E3.1 (easy) In the figure below, four different functions (a)-(d) are plotted with the constraints $0 \leq x \leq 2$.

(a) Which points in each graph are KKT-points with respect to minimization? Which points are local/global optima?

(b) The function in graph (a) is $f(x) = 2 - 0.5x$. Find all KKT-points analytically.

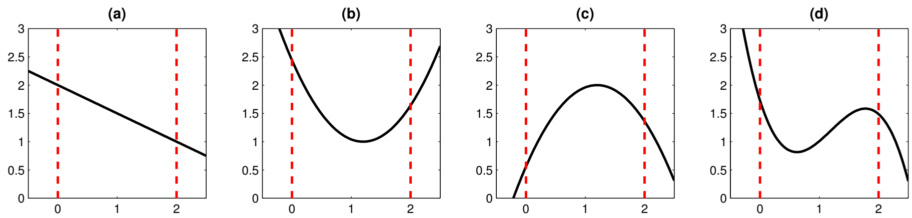

Figure 1: E3.1. Four different functions (a)-(d). All with constraints $0 \leq x \leq 2$.

E3.2 (easy) Consider the problem to

$$\begin{aligned} \text{minimize} \quad & e^{x_1} + x_1^2 x_2, \\ \text{subject to} \quad & x_1 + x_2^2 \geq 4, \\ & x_1, x_2 \geq 0. \end{aligned}$$

(a) State the KKT conditions for the problem.

(b) Are the KKT conditions satisfied at $(0, 2)^T$ and $(1, 1)^T$?

E3.3 (easy) In which of the following problems can we say that KKT conditions are necessary/sufficient for optimality?

(a)

$$\begin{aligned} \text{minimize} \quad & -2x_1 - 3x_2 + x_3, \\ \text{subject to} \quad & x_1 + 2x_2 + 2x_3 \leq 6, \\ & -6x_1 + 2x_2 - 2x_3 \geq 9, \\ & 2x_1 + 3x_2 + 5x_3 \leq 8, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(b)

$$\begin{aligned} \text{minimize} \quad & 2x_1^2 + 2x_2^2 - 4x_1 + x_1 x_2, \\ \text{subject to} \quad & x_1 + 2x_2 \leq 6, \\ & 2x_1 - 2x_2 \geq 2, \\ & x_1, x_2 \geq 0. \end{aligned}$$

(c)

$$\begin{array}{ll} \text{minimize} & 2x_1^2 - (x_2 - 1)^2 + 5, \\ \text{subject to} & x_1^2 + 2x_2^2 \leq 4, \\ & 3x_1 - x_2 \geq 1, \\ & x_1, x_2 \geq 0. \end{array}$$

(d)

$$\begin{array}{ll} \text{minimize} & x_1, \\ \text{subject to} & x_2 - (x_1 - 1)^3 \leq 0, \\ & x_1, x_2 \geq 0. \end{array}$$

E3.4 (medium) Consider the problem to

$$\begin{array}{ll} \text{minimize} & -x_1^3 + x_2^2 - 2x_1x_3^2, \\ \text{subject to} & 2x_1 + x_2^2 + x_3 = 5, \\ & 5x_1^2 - x_2^2 - x_3 \geq 2, \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

(a) State the KKT conditions for the problem.

(b) Verify that the KKT conditions are satisfied at $(1, 0, 3)^T$

E3.5 (medium) Consider the problem:

$$\begin{array}{ll} \text{minimize} & -2(x_1 - 2)^2 - x_2^2, \\ \text{subject to} & x_1^2 + x_2^2 \leq 25 \\ & x_1 \geq 0 \end{array}$$

(a) Does any constraint qualification hold?

(b) Find all KKT-points.

(c) Find the global minima.

E3.6 (medium) Consider the problem:

$$
\begin{aligned}
\text{minimize} \quad & 4x_1^2 + 2x_2^2 - 6x_1x_2 + x_1, \\
\text{subject to} \quad & -2x_1 + 2x_2 \geq 1, \\
& 2x_1 - x_2 \leq 0, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

Is $\boldsymbol{x} = (0, 1/2)^T$ a KKT point? Can you draw any conclusions from this regarding the optimality of $\boldsymbol{x}$?

E3.7 (easy) A problem where an objective function f should be minimized has been analyzed by Lagrangian relaxing some constraints and the dual function $q(\boldsymbol{\mu})$ has been created. An iterative algorithm has produced the following results.

(a) A primal feasible $\boldsymbol{x}^1$ has been found and $f(\boldsymbol{x}^1) = 6$. $\boldsymbol{\mu}^1$ is a vector with positive Lagrangian multipliers and $q(\boldsymbol{\mu}^1) = -2$. What can you say about the optimal objective value f^*?

(b) In the next iteration, two more vectors $\boldsymbol{x}^2$ and $\boldsymbol{\mu}^2$ (feasible) have been evaluated and $f(\boldsymbol{x}^2) = 3$ and $q(\boldsymbol{\mu}^2) = -4$. What can you now say about the optimal objective value f^*?

(c) Finally, we manage to maximize the dual function and $q^* = 3$. What are the conclusions regarding f^*?

E3.8 (easy) Consider the problem

$$
\begin{aligned}
\text{minimize} \quad & x_1^3x_2 - x_3^2 - x_1x_3, \\
\text{subject to} \quad & -x_1 + x_2 + x_3^2 \leq 4, \\
& x_1^3 - (x_2^2 - 1) + x_3 \geq 2, \\
& x_1 + x_3 \leq 6, \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

Lagrange relax the first two constraints and state the dual function q as a minimization problem.

E3.9 (medium) Consider the problem:

$$
\begin{aligned}
\text{minimize} \quad & x_1 - 3x_2, \\
\text{subject to} \quad & -x_1 + x_2 \leq 1, \\
& x_1 + x_2 \leq 4, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

Lagrangian relax the first constraint and create the Lagrange function $L(\boldsymbol{x}, \mu)$. Find and plot the Lagrangian dual function $q(\mu)$. What can you say about the optimal value of the primal problem? Can you find the optimal solution $\boldsymbol{x}^*$ to the primal problem?

E3.10 (easy) A company is producing loading pallets in two models, standard and extra long. Each model consists of three cross sectional beams with a length corresponding to the length of pallet. The standard model has five boards on the top and five at the bottom and takes 0.25 hours to assemble. The extra long model has nine boards on each side and takes 0.30 hours to assemble. The supplies of beams and boards are unlimited. It takes 0.005 hours to produce a beam for the standard model, 0.007 hours to produce a beam for extra long model, and 0.002 hours to produce a board. The company can sell any number of standard and extra long models with a profit of 50 SEK and 70 SEK, respectively. There are 200 hours available for assembly and 40 hours for production. Formulate the production planning to maximize the profit as an LP problem.
Formulate the Lagrangian dual problem that results from Lagrangian relaxing all but the sign constraints. State the global primal-dual optimality conditions.

E3.11 (medium) Consider the problem to

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}), \\ \text{subject to} & \mathbf{x} \in S, \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function, and S is some subset of $\mathbb{R}^n$. Assume that $\mathbf{x}^*$ satisfies the geometric optimality condition, and let γ be an arbitrary smooth curve through S starting at $\mathbf{x}^* \in S$, i.e., $\gamma : [0,1] \to S$ is a differentiable function with $\gamma(0) = \mathbf{x}^*$. Show that

$$\frac{d}{dt}\Big|_{t=0} f(\gamma(t)) \geq 0.$$