Main ideas and tools in the course in ODE

1. Integral form of I.V.P. to ODEs
2. Grönwall's inequality for showing uniqueness and continuity with respect to data.
3. Generalised eigenspaces of matrices. Basis of generalized eigenvectors.
4. Jordan form of matrices. Matrix functions, in particular exponent and logarithm.
5. Transfer matrix. Monodromy matrix.
6. Transfer mapping. Phase portrait.
7. Stability and instability of equilibrium points.
8. Linearization and Grobman - Hartman theorem. (iff $\mathrm{Re}(\lambda) \neq 0$)
9. Lyapunov functions (for stability, instability, and for finding positively invariant sets)
10. ω - limit sets. LaSalle's invariance principle for hunting ω - limit sets "living" in $V_f^{-1}(0)$.
11. Idea of solving integral equations by iterations (Banach's contraction prinile)

Examples of typical problems

Example on an application of Jordan matrix

For one particular solution of the system $\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t)$ with a real matrix A, the first component has the form $x_1 = t^2 + t\sin(t)$.

1. Which smallest size can the real matrix A have? **(4p)**

   Solution.

   The term $t\sin(t)$ in the solution is a sign that the Jordan form of the matrix A has a Jordan block corresponding to the eigenvalue $\lambda_1 = i$ that has multiplicity at least 2, for example $\begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$ or multiplicity 3 : $\begin{bmatrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{bmatrix}$ etc. On the other hand te matrix A is real and therefore it's characteristic plolynomial has real coefficients and therefore all complex eigenvalues must appear as conjugate pairs: the matrix A must have the eigenvalue $\lambda_2 = -i$ havingthe same multiplicity as λ_1, at least 2 and with corresponding Jordan block $\begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}$. The presence of the term t^2 in one component of a solution shows that the matrix A must have the eigenvalue $\lambda = 0$ with multiplicity at least 3 with correspoding Jordan block $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

All these observations imply that the real matrix A must have dimensions at least 7×7, because the sum of dimensions of sizes of Jordan blocks is at least $2 + 2 + 3 = 7$.■

Example of transition mapping.

1) Solve the initial value problem

$$\dot{x} = t\,x^3, \qquad x\,(1) = \xi$$

and find maximal intervals for solutions. Give a sketch of the domain for the transfer mapping $\varphi(t, 1, \xi){=}x(t)$ in the (t, x) plane.

2) Can one conclude which maximal interval have solutions to the similar equation

$$\dot{x} = t^3 x$$

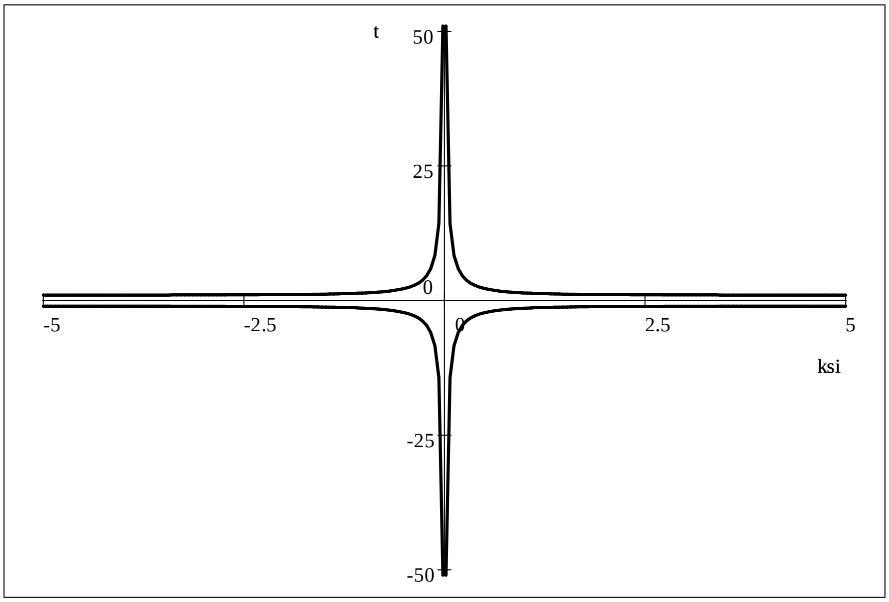
without solving it?

Solution.

1) It is the equation with separable variables.

$$
\begin{aligned}
\frac{dx}{dt} &= tx^3; \qquad x(1) = \xi \\
\int \frac{dx}{x^3} &= \int t dt \\
\frac{-1}{2x^2} &= \frac{t^2}{2} - C \\
C &= \frac{t^2}{2} + \frac{1}{2x^2}; \quad C = \frac{1}{2} + \frac{1}{2\xi^2} = \frac{1+\xi^2}{2\xi^2} \\
\frac{-1}{2x^2} &= \frac{t^2}{2} - \frac{1+\xi^2}{2\xi^2} \\
\frac{-1}{2x^2} &= \frac{\xi^2 t^2}{2\xi^2} - \frac{1+\xi^2}{2\xi^2} = \frac{\xi^2 t^2 - \left(1+\xi^2\right)}{2\xi^2} \\
x^2 &= \frac{\xi^2}{\left(1+\xi^2\right) - \xi^2 t^2} = \frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2} \\
x &= \sqrt{\frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2}}, \left(1+\xi^2\right)/\left(\xi^2\right) - t^2 > 0, \xi > 0 \\
x &= -\sqrt{\frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2}}, \left(1+\xi^2\right)/\left(\xi^2\right) - t^2 > 0, \xi < 0 \\
x &\equiv 0, \quad \xi = 0, \quad \textit{equilibrium} \\
\left(1+\xi^2\right)/\left(\xi^2\right) &> t^2; \quad t \in \left(-\sqrt{\left(1+\xi^2\right)/\left(\xi^2\right)}, \sqrt{\left(1+\xi^2\right)/\left(\xi^2\right)}\right) OPEN!!!
\end{aligned}
$$

1. The maximal intervals for these solutions are open in accordance with the general theory. One solution $x \equiv 0$ is defined on the whole $\mathbb{R}$. We draw boundaries of the domain for $\varphi(t, 1, \xi)$.



The equation $\dot{x} = t^3 x$ is defined on $\mathbb{R} \times \mathbb{R}$ and the right hand side satisfies on any compact time interval $[-R, R]$, $R > 0$ the estimate $\left|t^3 x\right| \leq R^3(1 + |x|)$ where the right hand side rises linearly with respect to $|x|$. It implies that the maximal existence interval for all solutions to this equation is $\mathbb{R}$.

Estimating Lyapunov functions V and their derivatives $V_f = \nabla V \cdot f$ along trajectories.

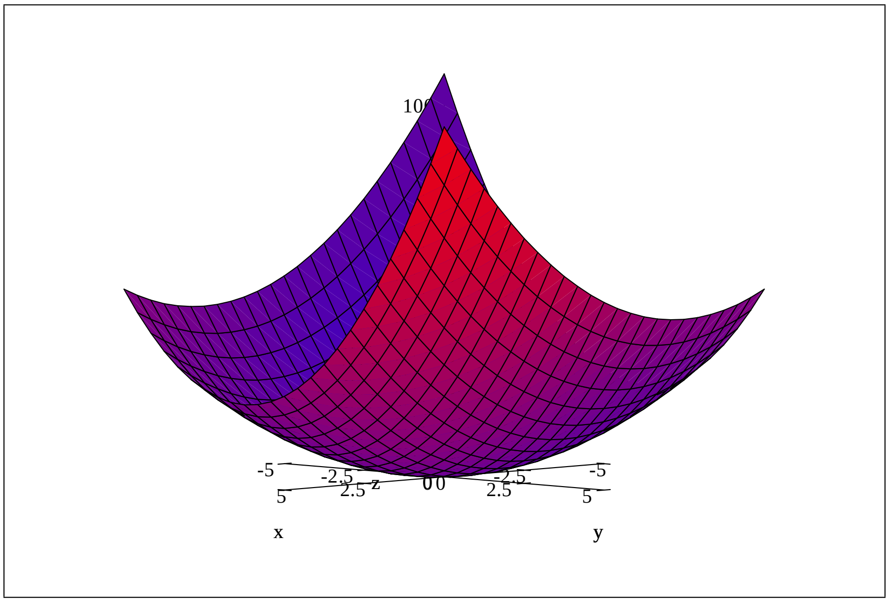
Investigation of positivity of functions V and $V_f = \nabla V \cdot f$.

Choosing a Lyapunov's function: it must be positive definite: $V(0) = 0$, $V(x) > 0$, $x \neq 0$.

Example: $V(x, y) = x^2 + xy + 2y^2$ Level sets of such a test function will be ellipces with the axis rotated with respect to the coordinate system. The inequality $|xy| \leq \frac{1}{2}\left(x^2 + y^2\right)$ helps to decide if the test function is positie definite.

This property is a condition in the stabiliuty theorem by Lyapunov, and it implies i particular that it's level sets are closed curves.

This property lets to use some of these level sets as boundaries for 1) positive invariant sets and 2) regions (or domains) of attraction for asymptotycally stable equilibrium points.



We like to have $V_f = \nabla V \cdot f(x)$ negative definite $V_f(x) < 0$ or at least $\nabla V \cdot f(x) \leq 0$ for $x \neq 0$. Here f is the right hand side ("velocity") in the differential equation of interest: $x' = f(x)$. It makes $\frac{d}{dt}(x(t)) = \nabla V \cdot f(x)-$ showing ow the test function changes along trajectories.

Check the set $V_f^{-1}(0)$ where $V(x) = 0$. Why?

The La Salle's invariance principle states that all ω - limit sets of trajectories inside the domain where $\nabla V \cdot f(x) \leq 0$ is valid, belong to the set $V_f^{-1}(0)$ and they belong even a smaller part of it that is the maximal invariant subset of $V_f^{-1}(0)$.

How to apply La Salle's invariance principle ?

i) The set $V_f^{-1}(0)$ is easy to identify, as a set of zeroes to V_f (in plane in most of our examples).

ii) The maximal invariant set inside $V_f^{-1}(0)$ (in the plane it will be a set of curves) it is easy to check invariant sets just looking on velocities $(f(x,y))$ on the set $V_f^{-1}(0)$ and checking if they go along curves forming $V_f^{-1}(0)$ or they go across.

Example.

Consider the following system of ODE: $\begin{cases} x' = & -x - 2y + xy^2 \\ y' = & 3x - 3y + y^3 \end{cases}$.

Show asymptotic stability of the equilibrium point in the origin and find the region of attraction for that.

Hint: applying Lyapunovs theorem, use the elementary inequality

$$|xy| \leq \frac{1}{2}\left(x^2 + y^2\right)$$

to estimate indefinite terms with xy.

A more general Young inequality can be useful for polynomials of higher degree in f:

$$|ab| \leq \frac{a^p}{p} + \frac{b^q}{q}; \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 1$$

Solution. Choose a test function $V(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$

$V_f = \nabla V \cdot f = x(-x-2y+xy^2)+y\left(3x-3y+y^3\right) = xy-x^2-3y^2+y^4+x^2y^2$ $= -x^2\left(1-y^2\right)-y^2\left(3-y^2\right)+\underset{indefinite\_term!}{xy} \leq -x^2\left(1-y^2\right)-y^2\left(3-y^2\right)+$ $0.5x^2+0.5y^2$

We apply the inequality $2xy \leq \left(x^2+y^2\right)$ to the last term and collecting terms with x^2 and y^2 arrive to the estimate

$V_f \leq -x^2\left(0.5-y^2\right)-y^2\left(2.5-y^2\right)$

It implies that $V_f < 0$ for $(x,y) \neq (0,0)$ and $|y| < 1/\sqrt{2}$.Therefore the Lyapunof function V is "strong" and therefore the origin is asymptotically stable.

The region of attraction is bounded by the largest levle set of V - a circle having the center in the origin that fits to the domain $|y| < 1/\sqrt{2}$, namely the circle: $\left(x^2+y^2\right) < 1/2$.

The second idea for choosing Lyapunov functions is choice of V **of polynomilas with arbitrary even powers and arbitrary coefficients.**

Another more clever choice of a test function as

$$V(x,y) = ax^m + by^n$$

in particular $V(x,y) = 3x^2+2y^2$ works in this particular case:

$V_f = 6x(-x-2y+xy^2)+4y(3x-3y+y^3) = 4y^4-12y^2-6x^2+6x^2y^2 = -4y^2\left(3-y^2\right)-6x^2\left(1-y^2\right) < 0$

for $|y| < 1$, therefore the ellipse $3x^2+2y^2 < 2$ that fits into the stripe $|y| < 1$ in the plane is a region of attraction for the asymptotically stable equilibrium in the origin.

One can also observe the asymptotic stability of the origin here by linearization with variational matrix

$A = \begin{bmatrix} -1 & -2 \\ 3 & -3 \end{bmatrix}$, with characteristic polynomial: $\lambda^2+4\lambda+9 = 0$, and calculating eigenvalues: $-i\sqrt{5}-2, i\sqrt{5}-2$ with $\operatorname{Re}\lambda < 0$. But linearization gives no information about the domain of attraction.

Application of Poincare - Bendixson theorem

The generalized Poincare-Bendixson's theorem gives a complete description of possible types of ω **- limit sets in the plane** $\mathbb{R}^2$.

Theorem (generalized Poincare-Bendixson)

Let M be an open subset of $\mathbb{R}^2$ and $f: M \to \mathbb{R}^2$ and $f \in C^1$. Fix $\xi \in M$ and suppose that $\Omega(\xi) \neq \emptyset$, compact, connected and contains only finitely many equilibrium points.

Then one of the following cases holds:

(i) $\Omega(\xi)$ is an equilibrium point

(ii) $\Omega(\xi)$ is a periodic orbit

(iii) $\Omega(\xi)$ consists of finitely many fixed points $\{x_j\}$and non-closed orbits γ such that ω and α - limit points of γ belong to $\{x_j\}$.

1. Consider the following system of ODEs. $\begin{cases} x' = & y \\ y' = & -x - y\left[\ln\left(x^2+4y^2\right)\right] \end{cases}$.

   Show that this system has a non-trivial periodic solution. **(4p)**

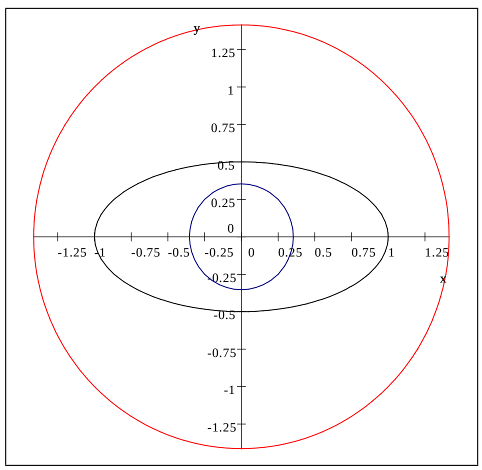
   Point out that the origin is outside the domain of the equation.

   Solution.

   Consider the test function $E(x,y) = \frac{1}{2}\left(x^2+y^2\right)$

   $\frac{d}{dt}E(u(t),v(t)) = \nabla E \cdot f(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} y \\ -x - y\left[\ln\left(x^2+4y^2\right)\right] \end{bmatrix} = -y^2\left[\ln\left(x^2+4y^2\right)\right] \begin{cases} \geq 0 & 0 < x^2 - \\ \leq 0 & x^2 - \end{cases}$

   The boundary curve $x^2 + 4y^2 = 1$



   is the ellipse with halv axes 1 and 1/2 i the x - direction with center in the origin.

   Therefore any circle with the center in the origin inside this ellipse is never entered by a trajectory.

   Similarly any circle with the center in the origin autside this ellipse is never left by a trajectory.

   Such two circles build an annulus that is a positively invariant set for this system of ODEs.

   For example an annulus $1/4 \leq x^2 + y^2 \leq 1$ satisfies this conditions. This annulus contains no equilibrium points, because the origin is the only equilibrium point. Therefore by Poincare - Bendixson theorem this annulus must contain at least one periodic orbit.■

Poincare - Bendixson theorem and testing the absence of equilibrium points in a positive invariant set.

We try to find an ring shaped domain that is positively invariant and need to check three conditions:

i) The outer boundary of the ring (using a level set of a test function, or a polygon shaped domain testing velosities on each segment of it's boundary)

ii) The inner boundary of the ring (using a level set of a test function, or linearization for identifying a repeller inside a large postively invariant set by applying the Grobman - Hartman theorem)

iii) Check that no equilibrium points exist inside of the ring (is missed often by students)

Example. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = x - 2y - x\left(2x^2 + y^2\right) \\ y' = 4x + y - y\left(2x^2 + y^2\right) \end{cases} \qquad \textbf{(4p)}$$

Solution. Consider the following test function: $V(x,y) = 2x^2 + y^2$. Denoting the right hand side in the equation by vectorfunction $F(x,y)$ we conclude that

$V_f = \nabla V \cdot f = 4x^2 - 8xy - 4x^2\left(2x^2 + y^2\right) + 8xy + 2y^2 - 2y^2(2x^2 + y) = 2\left(1 - (2x^2 + y^2)\right)(2x^2 + y^2)$.

It implies that the elliptic shaped ring: $R = \left\{(x,y) : 0.5 \leq (2x^2 + y) \leq 2\right\}$ is a positive invariant compact set for the ODE, because velocity vectors are directed inside of this ring both on it's outer and inner boundaries ($\nabla V \cdot F < 0$ for $(2x^2 + y) = 2$ and $\nabla V \cdot F > 0$ for $(2x^2 + y) = 0.5$.

The origin is the only equilibrium point of the system**. It is not so easy to see from the system of equations itself.** But one can see it easier by cheching first zeroes of $V_f(x,y)$ that is a scalar function and evidently must be zero in all equilibrium points..

We observe that $V(x,y) = 2x^2 + y^2$ is positive definite and $\nabla V \cdot f(x,y) = 0$ only if $(x,y) = (0,0)$ or if $(2x^2+y^2) = 1$.But it is easy to see from the expression for the right hand side for the ODE that in the last case (x,y) cannot be equilibrium point because the right hand side becomes linear with nondegenerate matrix and is zero only in the origin $(x,y) = (0,0)$. The equation for equilibrium points on the level set $(2x^2 + y^2) = 1$ is the following:

$$\begin{cases} 0 = x - 2y - x = -2y \\ 0 = 4x + y - y = 4x \end{cases}$$

By the Poincare-Bendixson theorem the positively invariant set R not including any equilibrium point must include at least one orbit of a periodic solution.■

Problem on ω - limit sets(January 2020)

Consider the following system of ODEs. $\begin{cases} x' = & y \\ y' = & x - x^3 - ay\left(y^2 - x^2 + \frac{1}{2}x^4\right), \quad a > 0 \end{cases}$.

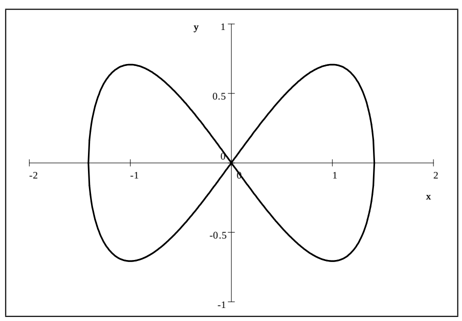
1. Find all systems equilibrium points. Show using the test function $H = \frac{1}{2}\left(y^2 - x^2 + \frac{1}{2}x^4\right)$ and La Salle's invariance principle, that the level set $H(x, y) = 0$ includes ω - limit sets of this system for all points in the plane except a finite number. Sketch these ω - limit sets. **(4p)**

Solution.

The system has three equilibrium points, all on the $x-$axis: $(-1, 0)$, $(0, 0)$, $(1, 0)$. The level set $H(x, y) = \frac{1}{2}\left(y^2 - x^2 + \frac{1}{2}x^4\right) = 0$ has the shape of ∞ with the center in the origin. One can see it by expressing y in terms of x:

$$y = \pm |x| \sqrt{1 - \frac{1}{2}x^2}$$

The ∞ figure is symmetric with respect to x - axis and cuts it in points $\pm\sqrt{2}$. The formula above implies that $H(x, y) > 0$ outside of the ∞ figure, and $H(x, y) < 0$ inside of the ∞ figure.



We calculate how the H function changes along trajectories.

$$H_f(x, y) = \frac{d}{dt}H(x(t), y(t)) = \begin{bmatrix} -x + x^3 \\ y \end{bmatrix} \cdot \begin{bmatrix} y \\ x - x^3 - ay\left(y^2 - x^2 + \frac{1}{2}x^4\right) \end{bmatrix} =$$

$$\underbrace{-xy + x^3y + xy - x^3y}_{=0} - ay^2 \underbrace{\left(y^2 - x^2 + \frac{1}{2}x^4\right)}_{H(x,y)}$$

We point out that $\frac{d}{dt}H(x(t), y(t)) = 0$ on the level set $H(x, y) = 0$ (the ∞ figure) and on the x - axis. It means that trajectories are tangential to the level set $H(x, y) = 0$. Therefore ∞ - figure is an invariant set for the system and consists of three orbits: the equilibrium in the origin (that is a saddle point, easily seen by linerization) and two closed branches of the ∞ figure correspoding to $x > 0$ and $x < 0$ in the expression $y = \pm |x| \sqrt{1 - \frac{1}{2}x^2}$.

$H_f(x, y) = \frac{d}{dt}H(x(t), y(t)) < 0$ outside of the ∞ figure and not on the x - axis where $\frac{d}{dt}H(x(t), y(t)) = 0$.

$H_f(x,y) = \frac{d}{dt}H(x(t),y(t)) > 0$ inside of the ∞ figure and not on the x - axis where $\frac{d}{dt}H(x(t),y(t)) = 0$.

By La Salle's invariance principle all trajectories are attracted to the largest invariat set inside the set $H_f^{-1}(0)$,were $H_f(x,y) = 0$. This set consists of the union of the ∞ figure and the x - axis. There are no invariant sets on the x - axis except three equilibrium points $(-1,0)$, $(0,0)$, $(1,0)$.

It implies that for all points in the plain except equilibrium points, and points on the ∞ figure, $H(x(t),y(t))$ tends to zero along trajectories. The ω - limit sets for these points consist of one of the branches of the ∞ figure (for points inside it) or of the whole ∞ figure - for points outside it. The origin is the ω - limit set for all points on the ∞ figure. Equilibrium points are ω - limit sets of themselfs

Problem on stability of equilibrium points and on domains of attraction.

Consider the following system of ODEs. $\begin{cases} x' = & 1 - xy \\ y' = & x - y^3 \end{cases}$

Find all equilibrium points and investigate their stability. Find domains of attraction for possible asymptotically stable equilibrium points. **(4p)**

Solution.

Equilibrium points are $(1,1)$ and $(-1,-1)$ can be found by substitution. $x = y^3$, $1 = xy = y^4$.

Jacoby matrix of the right hand side is $J(x,y) = \begin{bmatrix} -y & -x \\ 1 & -3y^2 \end{bmatrix}$; $J(1,1) = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$; $J(-1,-1) = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$. $\det(J(1,1)) = 4$, $\operatorname{tr}(J(1,1)) = -4$. Therefore the equilibrium point $(1,1)$ is asymptotically stable.

$\det(J(-1,-1)) = -4$. Therefore the linearized around $(-1,-1)$ system has a saddle point and the equilibrium point $(-1,-1)$ is unstable.

We shift the origin of the coordinate system into the point $(1,1)$ by introducing new variables $u = x-1, v = y-1$ and $x = u+1$, $y = v+1$.

$$\begin{cases} u' = & -u - v - uv \\ v' = & u - 3v - 3v^2 - v^3 \end{cases}$$

Consider a test function $E(u,v) = \frac{1}{2}\left(u^2+v^2\right)$

$$\begin{aligned} \frac{d}{dt}E(u(t),v(t)) &= \begin{bmatrix} u \\ v \end{bmatrix} \cdot \begin{bmatrix} -u - v - uv \\ u - 3v - 3v^2 - v^3 \end{bmatrix} = \\ &= -u^2 - uv - u^2v + uv - 3v^2 - 3v^3 - v^4 = \\ &= -u^2(1-v) - 3v^2\underbrace{(1+v+v^2)}_{>0} < 0 \end{aligned}$$

$$if \quad v < 1, \quad (u,v) \neq (0,0)$$

The largest circle in (u, v) plane satisfying the condition $v \leq 1$ has radius 1. Therefore the circle of radius 1 around the equilibrium point $(1, 1)$ is the domain of attraction for the asymptotically stable equilibrium $(1, 1)$ of the original system of ODEs.■