Class Lectures (for Chapter 7)

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Deeper aspects of measure theory.

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We will need a lot of preliminary work, including the so-called Hahn and Jordan Decomposition theorems.

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c. Condition (ii) is there to avoid having $\infty - \infty$.

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This is the picture we want in general.

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Theorem

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Then a Hahn decomposition is given by $([0, \frac{3}{4}], (\frac{3}{4}, 1])$.

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Read lecture notes.

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If we can show that P^c is a negative set, we would be done.

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Two measures μ and ν on $(X, \mathcal{M})$ are **mutually singular** , $\mu \perp \nu$, if X can be partitioned into two disjoint sets E and F in $\mathcal{M}$ so that $\mu(E) = 0 = \nu(F)$. So μ "lives on F" and ν "lives on E". (For signed measures, the definition needs to be modified.)

Example: Let $X = [0, 1]$ with the Borel sets.

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Example: The Cantor measure and Lebesgue measure.

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Example: The Cantor measure and Lebesgue measure. $E = C$ and $F = C^c$.

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No. μ_1 and μ_2 are not mutually singular.

Instead one should take ν^+ to be Lebesgue measure restricted to $[0, 1/4]$ and ν^- to be Lebesgue measure restricted to $[3/4, 1]$.

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$\nu^+(N) = 0 = \nu^-(P)$ and so $\nu^+ \perp \nu^-$.
Also

$$(\nu^+ - \nu^-)(A) = \nu^+(A) - \nu^-(A) = \nu(A \cap P) + \nu(A \cap N) = \nu(A).$$

QED

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(Convince yourself this is a measure; uses linearity of the integral and the Monotone Convergence Theorem.) ν is called $f\mu$ and one has $\nu \ll \mu$.

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Let ν *and* μ *be two measures on* $(X, \mathcal{M})$ *with* $\nu \ll \mu$ *and with* ν *and* μ *being* σ*-finite.*

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This is false if one does not assume σ-finiteness.

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3. (Kolmogorov) The Radon-Nikodym Theorem is crucial in advanced probability when one deals with the subtle concept of conditioning.

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Let $X = \{1, 2, 3\}$ (full σ-algebra).

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$$\mathcal{F} := \{f : X \to [0, \infty) : \int_A f(x) d\mu(x) \le \nu(A) \ \ \forall A \in \mathcal{M}\}.$$

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claim: There exists $f_0 \in \mathcal{F}$ for which $\int f_0(x) d\mu(x) = m$; i.e. the supremum above is achieved.

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we have that (1) each $g_n \in \mathcal{F}$ from the subclaim,

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Proof of The Radon-Nikodym Theorem for finite measure spaces

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This f_0 will turn out to be our Radon Nikodym derivative.

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The Lebesgue Decomposition Theorem

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The Lebesgue Decomposition for a simple example

Let $X = \{1, 2, 3\}$ (full σ-algebra).

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Also what is the Radon Nikodym derivative of ν_{ac} with respect to μ? The function $(0, 0, 2)$. Or in fact $(0, x, 2)$ for any value of x since this is just a change on a set of μ measure 0.

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One shows that $g_0 := f_0 + \epsilon_n I_{P_n} \in \mathcal{F}$ and $\int g_0 d\mu(x) = m + \epsilon_n \mu(P_n) > m$, a contradiction.

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Then $\mu|_{\mathcal{A}}$ is atomic, $\mu|_{\mathcal{A}^c}$ is continuous and these measures are mutually singular.

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The exact same theorem and proof works in R^n with n-dimensional Lebesgue measure.

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Now $n \to \infty$ using continuity from above for ν (ν is a finite measure) gives $\nu(A) \geq \epsilon_0$.
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