Chalmers/Gothenburg University
Mathematical Sciences

EXAM SOLUTION

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date: 09–08–27

Examiner: Michael Patriksson

Question 1

(the simplex method)

(2p) a) To transform the problem to standard form, first change the sign on the second contraint and then add a non-negative slack variable to the first constraint and subtract a non-negative slack (surplus) variable from the second. We get

$$\begin{array}{lrl} \text{minimize} & z = & 2x_1 - x_2 + x_3, \\ \text{subject to} & & x_1 + 2x_2 - x_3 + s_1 = 7, \\ & & 2x_1 - x_2 + 3x_3 - s_2 = 3, \\ & & x_1, \quad x_2 \quad x_3, \quad s_1, \quad s_2 \geq 0. \end{array}$$

Now start phase 1 using an artificial variable $a \geq 0$ added in the second constraint. s_1 can be used as a second basic variable.

$$\begin{array}{lrl} \text{minimize} & w = & a, \\ \text{subject to} & & x_1 + 2x_2 - x_3 + s_1 = 7, \\ & & 2x_1 - x_2 + 3x_3 - s_2 + a = 3, \\ & & x_1, \quad x_2 \quad x_3, \quad s_1, \; s_2, \quad a \geq 0. \end{array}$$

We start with the BFS given by $(s_1, a)^{\mathrm{T}}$. In the first iteration of the simplex algorithm, x_3 has the least reduced cost (-3) and is chosen as the incoming variable. The minimum ratio test then shows that a should leave the basis. By updating the basis and computing the reduced costs we see that we are now optimal with $w^* = 0$ and we proceed to phase 2.

The BFS is given by $\boldsymbol{x}_B = (s_1, x_3)^{\mathrm{T}}$, $\boldsymbol{x}_N = (x_1, x_2, s_2)^{\mathrm{T}}$ and the reduced costs with the phase 2 cost vector are $\tilde{\boldsymbol{c}}^{\mathrm{T}}_{(x_1,x_2,s_2)} = (\frac{4}{3}, -\frac{2}{3}, \frac{1}{3})$. The reduced cost is negative for x_2 which is the only eligable incoming variable. $\boldsymbol{B}^{-1}\boldsymbol{b} = (8, 1)^{\mathrm{T}}$ and $\boldsymbol{B}^{-1}\boldsymbol{N}_{x_2} = (\frac{5}{3}, -\frac{1}{3})^{\mathrm{T}}$, so the minimum ratio test shows that s_1 should leave the basis. Updating the basis and computing the new reduced costs gives $\tilde{\boldsymbol{c}}^{\mathrm{T}}_{(x_1,s_1,s_2)} = (2, \frac{2}{5}, \frac{1}{5}) \geq \boldsymbol{0}$ and thus the optimality condition is fulfilled for the current basis. We have $\boldsymbol{x}^*_B = (\frac{24}{5}, \frac{13}{5})^{\mathrm{T}}$, or in the original variables, $\boldsymbol{x}^* = (x_1, x_2, x_3)^* = (0, \frac{24}{5}, \frac{13}{5})^{\mathrm{T}}$, with the optimal value $z^* = -\frac{11}{5}$.

(1p) b) The reduced costs are not affected by the right-hand-side vector, so the only thing that has to be checked is when the current basis stays feasible.

$$\boldsymbol{B}^{-1}\boldsymbol{b} \geq \mathbf{0} \Leftrightarrow \frac{1}{3}\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ 3 \end{pmatrix} \geq \mathbf{0} \Leftrightarrow \begin{cases} b_1 + 1 \geq 0 \\ 3 \qquad \geq 0 \end{cases} \Leftrightarrow b_1 \geq -1$$

Thus, the current basis stays optimal for all $b_1 \geq -1$.

(3p) Question 2

(modeling)

Introduce the variables x_{ij} = number of workhours that the crew j spends in turbine i,

$$y_{ij} = \begin{cases} 1, & \text{crew } j \text{ performs maintenance at turbine } i, \\ 0, & \text{otherwise;} \end{cases}$$

$$z_i = \begin{cases} 1, & \text{the turbine } i \text{ is not operational,} \\ 0, & \text{otherwise.} \end{cases}$$

The model is

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} z_i e_i \\
\text{subject to} \quad & d_i - \sum_{j=1}^{m} x_{ij} \leq d_i z_i, \quad i \in \{1, \dots, n\}, \\
& x_{ij} \leq d_j y_{ij}, \quad i \in \{1, \dots, n\}, j \in \{1, \dots, m\}, \\
& \sum_{i=1}^{n} y_{ij} \leq 2, \quad i \in \{1, \dots, n\}, j \in \{1, \dots, m\}, \\
& \sum_{i=1}^{n} x_{ij} + \sum_{i=1}^{n} 2c_{ij} y_{ij} \leq 8, \quad j \in \{1, \dots, m\}, \\
& x_{ij} \geq 0, y_{ij}, z_i \in \{0, 1\} \quad i \in \{1, \dots, n\}, j \in \{1, \dots, m\}.
\end{aligned}$$

Question 3

(optimality conditions)

See The Book, Theorem 10.10.

Question 4

(exterior penalty method)

(1p) a) Direct application of the KKT conditions yield that $\boldsymbol{x}^* = (\frac{3}{5}, \frac{2}{5})^{\mathrm{T}}$ and $\lambda^* = -1/5$ uniquely.

(1p) b) Letting the penalty parameter be $\nu > 0$, it follows that $\boldsymbol{x}_\nu = \frac{\nu}{1+5\nu}(3,2)^{\mathrm{T}}$. Clearly, as $\nu \to \infty$ convergence to the optimal primal–dual solution follows.

(1p) c) From the stationarity conditions of the penalty function $\boldsymbol{x} \mapsto f(\boldsymbol{x}) + \lambda h(\boldsymbol{x}) + \nu |h(\boldsymbol{x})|^2$ follow that $\boldsymbol{x}_\nu$ fulfills $\nabla f(\boldsymbol{x}_\nu) + [2\nu h(\boldsymbol{x}_\nu)]\nabla h(\boldsymbol{x}_\nu) = 0^2$, and hence a proper Lagrange multiplier estimate comes out as $\lambda_\nu := 2\nu h(\boldsymbol{x}_\nu)$. Insertion from b) yields $\lambda_\nu = \frac{-\nu}{1+5\nu}$, which tends to $\lambda^* = -\frac{1}{5}$ as $\nu \to \infty$.

Question 5

(topics in convexity)

(2p) a) See Theorem 3.40.

(1p) b) See Theorem 3.42.

(3p) Question 6

(Lagrangian dual)

$L(x,\mu) = -x_1 - 1/2x_2 + \mu_1(x_1^2 + x_2^2 - 1) + \mu_2(1 - (x_1-1)^2 - (x_2-1)^2)$. The dual function is $q(\mu) = \min_x(L(x,\mu)) = \min_{x_1} \underbrace{(-x_1 + \mu_1 x_1^2 - \mu_2(x_1-1)^2)}_{q_1(x_1)}$

$+ \min_{x_2} \underbrace{(-1/2x_1 + \mu_1 x_1^2 - \mu_2(x_1-1)^2) + \mu_2}_{q_2(x_2)}.$

$\frac{dq_1}{dx_1} = -1+2\mu_1 x_1 - 2\mu_2(x_1-1)$ and $\frac{d^2q_1}{dx_1^2} = 2(\mu_1-\mu_2)$. We notice that q_1 is strictly convex for $\mu_1 > \mu_2$ and strictly concave for $\mu_1 < \mu_2$ and linear for $\mu_1 = \mu_2$. For $\mu_1 > \mu_2$ the minimum is attained attained at $x_1 = \frac{1-2\mu_2}{2(\mu_1-\mu_2)}$ and is $-\infty$ for $\mu_1 < \mu_2$. Similarly for q_2 we obtain $x_2 = \frac{1/2-2\mu_2}{2(\mu_1-\mu_2)}$. Simplifying and inserting into L yields $q(\mu) = \frac{8(3-2\mu_2)\mu_2-16\mu_1^2-5}{16(\mu_1-\mu_2)}$ if $\mu_1 > \mu_2$. If $\mu_1 = \mu_2$ the derivatives of q_1 and q_2 can not be zero simultaneosly. We therefore have $q_1(\mu) = -\infty$ or $q_2(\mu) = -\infty$. We therefore have $q(\mu) = -\infty$ if $\mu_1 \leq \mu_2$.

The dual problem can be formulated as $\max_{\mu \geq 0} q(\mu)$. The dual problem is always convex; in the pressent case it is also differentiable.

$q(1, 1/2) = -13/8$ and $f(0,1) = -1/2$; we can therefore conclude (by weak duality) that $-13/8 \leq f^* \leq -1/2$.

Drawing the feasible region together with the linear objective gives the optimal solution $x^* = (1,0)$, $f^* = -1$.

The problem is non-convex, hence a dual gap can exist. Assume there is no duality gap, then according to Theorem 6.7 $L(x^*, \mu^*) = \min_x L(x, \mu^*)$. If μ^* is optimal then $\mu_1^* > \mu_2^*$. Since the function $L(\cdot, \mu)$ is strictly convex, the minimum is obtained at $\nabla_x L(\cdot, \mu) = 0$. Therefore $1 = \frac{1-2\mu_2}{2(\mu_1-\mu_2)}$ and $0 = \frac{1/2-2\mu_2}{2(\mu_1-\mu_2)}$. This yields $\mu_2 = 1/4$ and $\mu_1 = 1/2$. Since $q(1/2, 1/4) = -1$ no duality gap exists.

Question 7

(true or false claims in optimization)

(1p) a) True. The important implication is that if a problem is unbounded, then its dual must be infeasible. The adding of an extra variable relaxes the original problem. Since there is a feasible point to the original problem, the extended problem will also have a feasible solution (e.g., by setting $x_4 = 0$). If the dual to the extended problem is unbounded the primal problem (dual to the dual) must be infeasible. This is not the case and the claim is proved.

(1p) b) True. The equality subsystem at $(1,1,1)^{\mathrm{T}}$ consists of all rows but the third, so

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The rank of $\tilde{A}$ is 3 since the first three rows are linearly independent. So, $rank(\tilde{A}) = n$ which implies that the proposed point is an extreme point (in this case corresponding to a degenerate basis).

(1p) c) False. A counterexample in $\mathbb{R}^2$ is given by the problem defined by $f(\boldsymbol{x}) = x_2$, $g(\boldsymbol{x}) = -x_1^2 - x_2$ at the point $\boldsymbol{x}^* = (0, 0)^{\mathrm{T}}$. The conditions are fulfilled, but all balls around $\boldsymbol{x}^*$ contain points with smaller objective values.