Class Lectures (for Chapter 8)

Theory of Differentiation in R^n: Overview

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Weierstrass then shocked the community when he constructed a continuous nowhere differentiable function on $[0, 1]$. (Almost all continuous functions are nowhere differentiable).

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Since $|g| \leq 1$ and is continuous, f is continuous by the Weierstrass M-Test (or some other test from advanced calculus). But, not worrying about being rigorous, we have that

$$f'(x) := \sum_n \frac{4^n}{3^n} g'(4^n(x))$$

whose terms go to ∞.

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$$P(f \text{ is nowhere differentiable}) = 1$$

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2. For the Cantor ternary function, we have seen that the derivative is 0 a.e.

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Most of the rest of the course are generalizations of the fundamental theorems of calculus.

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have, by "First", the same derivative $(f'(x))$ and agree at a. Hence equal.

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This is the general case.

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$$\lim_{r \to 0} \frac{\int_x^{x+r} f(t) dt}{r} = \lim_{r \to 0} \frac{F(x+r) - F(x)}{r} = f(x) \text{ for a.e. } x$$

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For the Cantor Ternary function, we have seen the measure is singular w.r.t. m.

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Remarks: We have seen in an exercise that

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So you can approximate sets "from the outside" by open sets and "from the inside" by compact sets.

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The irrationals $[0,1]\backslash Q$ cannot be approximated "from the inside" by open sets, since it contains no nonempty open sets. Similarly the rationals $[0,1] \cap Q$ cannot be approximated "from the outside" by closed sets, since $[0,1]$ is the only closed set containing the rationals.

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Let ν be a regular Borel measure on R^n. If E is a Borel set, then

$$\nu(E) = \inf\{\nu(O) : O \supseteq E, O \text{ open }\}.$$

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$$\sum_{i=1}^{k} m(B_i) \geq \frac{c}{3^n}.$$

Proof of the covering lemma

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(1) implies that $K \subseteq \bigcup_{j=1}^{k} B_j^*$ which in turn yields

$$c < m(K) \leq m(\bigcup_{j=1}^{k} B_j^*) \leq \sum_{j=1}^{k} m(B_j^*) = 3^n \sum_{j=1}^{k} m(B_j).$$

QED

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2. We will use the believable fact that $A_r f(x)$ is a continuous function of (r, x) on $(0, \infty) \times R^n$.

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where the last inequality follows from the disjointness of the balls. Since this is true for all $c < m(E_\alpha)$, we are done.
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$$m(E_\alpha) \le m(\{x : H(f-g)(x) > \frac{\alpha}{2}\}) + m(\{x : |g(x) - f(x)| > \frac{\alpha}{2}\}).$$

Use Maximal inequality and Markov's inequality, we get

$$m(E_\alpha) \le \frac{2(3^n)}{\alpha}\epsilon + \frac{2}{\alpha}\epsilon.$$

Since this is true for every $\epsilon > 0$, we can conclude $m(E_\alpha) = 0$.
QED

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(The set of x where this holds is called the **Lebesgue set** *of f and is denoted by L_f.)*

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Since this inequality holds for all $q \in Q$, the LHS is 0. QED

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Example: In R^2, $E_r := [0, \frac{r}{2}] \times [0, \frac{r}{200}]$ shrinks nicely to $(0,0)$ but $E_r := [0, \frac{r}{2}] \times [0, \frac{r^2}{2}]$ does not shrink nicely to $(0,0)$.

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QED

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However, this still doesn't give us Lebesgue's Theorem that a monotone function is differentiable a.e.

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We leave this to read on your own as it gets quite technical.

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Theorem

(Lebesgue) If $f : [0,1] \to R$ is monotone ($x \leq y$ implies that $f(x) \leq f(y)$), then for a.e. x, f is differentiable with a finite derivative.

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$$\frac{f(x+h)-f(x)}{h}=\frac{\mu_f((x,x+h])}{m((x,x+h])}$$

converges a.e.

We even know what the limit is a.e. Namely, writing the Lebesgue decomposition of μ_f with respect m

$$\mu_f=\mu_s+gdm$$

the limit is $g(x)$ a.e. So, the derivative of f is the Radon-Nikodym derivative of the the absolute continuous part of μ_f wrt m.

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Examples: (1). f is the Cantor ternary function. (2). f is continuously differentiable. We will see that $\mu_f \ll m$ and f' is the RN derivative wrt m. (Elementary probability.)

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Theorem

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