

Class Lectures (for Chapter 8)

Theory of Differentiation in R^n : Overview

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Weierstrass then shocked the community when he constructed a continuous nowhere differentiable function on $[0, 1]$. (Almost all continuous functions are nowhere differentiable).

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Since $|g| \leq 1$ and is continuous, f is continuous by the Weierstrass M-Test (or some other test from advanced calculus). But, not worrying about being rigorous, we have that

$$f'(x) := \sum_n \frac{4^n}{3^n} g'(4^n(x))$$

whose terms go to ∞ .

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$$P(f \text{ is nowhere differentiable}) = 1$$

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Most of the rest of the course are generalizations of the fundamental theorems of calculus.

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have, by "First", the same derivative ($f'(x)$) and agree at a . Hence equal.

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This is the general case.

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We will even obtain a higher dimension analogue called Lebesgue's Differentiation Theorem. However, this will not give us Lebesgue's Theorem. For that, we will need a version of Lebesgue's Differentiation Theorem for measures.

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This holds if and only if the measure μ_f , associated to f , is absolutely continuous with respect to m .

For the Cantor Ternary function, we have seen the measure is singular w.r.t. m .

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Remarks: We have seen in an exercise that

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So you can approximate sets "from the outside" by open sets and "from the inside" by compact sets.

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The irrationals $[0, 1] \setminus \mathbb{Q}$ cannot be approximated "from the inside" by open sets, since it contains no nonempty open sets. Similarly the rationals $[0, 1] \cap \mathbb{Q}$ cannot be approximated "from the outside" by closed sets, since $[0, 1]$ is the only closed set containing the rationals.

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Let ν be a regular Borel measure on R^n . If E is a Borel set, then

$$\nu(E) = \inf\{\nu(O) : O \supseteq E, O \text{ open}\}.$$

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Let \mathcal{C} be a collection of open balls in R^n with U being their union. For all $c < m(U)$, there exist $B_1, B_2, \dots, B_k \in \mathcal{C}$ which are disjoint satisfying

$$\sum_{i=1}^k m(B_i) \geq \frac{c}{3^n}.$$

Proof of the covering lemma

Choose $K \subseteq U$ compact so that $m(K) > c$

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Choose $K \subseteq U$ compact so that $m(K) > c$ and by compactness, choose $A_1, \dots, A_m \in \mathcal{C}$ which cover K .

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(1) implies that $K \subseteq \bigcup_{j=1}^k B_j^*$ which in turn yields

$$c < m(K) \leq m\left(\bigcup_{j=1}^k B_j^*\right) \leq \sum_{j=1}^k m(B_j^*) = 3^n \sum_{j=1}^k m(B_j).$$

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2. We will use the believable fact that $A_r f(x)$ is a continuous function of (r, x) on $(0, \infty) \times \mathbb{R}^n$.

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Use Maximal inequality and Markov's inequality, we get

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$$E_\alpha \subseteq \{x : H(f - g)(x) > \frac{\alpha}{2}\} \cup \{x : |g(x) - f(x)| > \frac{\alpha}{2}\}.$$

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*(The set of x where this holds is called the **Lebesgue set** of f and is denoted by L_f .)*

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Since this inequality holds for all $q \in Q$, the LHS is 0. QED

Nicely shrinking sets

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Example: In R^2 , $E_r := [0, \frac{r}{2}] \times [0, \frac{r}{200}]$ shrinks nicely to $(0, 0)$ but $E_r := [0, \frac{r}{2}] \times [0, \frac{r^2}{2}]$ does not shrink nicely to $(0, 0)$.

Lebesgue's Theorem for nicely shrinking set

Corollary

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However, this still doesn't give us Lebesgue's Theorem that a monotone function is differentiable a.e.

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We leave this to read on your own as it gets quite technical.

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Theorem

(Lebesgue) If $f : [0, 1] \rightarrow R$ is monotone ($x \leq y$ implies that $f(x) \leq f(y)$), then for a.e. x , f is differentiable with a finite derivative.

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$$\mu_f = \mu_s + gdm$$

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$$\frac{f(x+h) - f(x)}{h} = \frac{\mu_f((x, x+h])}{m((x, x+h])}$$

converges a.e.

We even know what the limit is a.e. Namely, writing the Lebesgue decomposition of μ_f with respect m

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Examples: (1). f is the Cantor ternary function. (2). f is continuously differentiable. We will see that $\mu_f \ll m$ and f' is the RN derivative wrt m . (Elementary probability.)

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