Exercise 1. A sequence \((a_k)_{k=1}^{\infty}\) converges absolutely if \(\sum_{k=1}^{\infty} |a_k| < \infty\). In this case \(\sum_{k} a_k\) exists and is independent of summation order, i.e. \(\sum_{k} a_k = \lim_{j \to \infty} \sum_{k=1}^{j} a_{n_j}\) for any enumeration \(\{n_j\}_{j=1}^{\infty}\) of \(\mathbb{N}\). Show that this claim follows from DCT.

Exercise 2. Given \(r < 1\) and \(\Theta \in [-\pi, \pi)\), show that \(\sum_{k=-\infty}^{\infty} r^{|k|} e^{i k \Theta}\) converges absolutely and equals \(\frac{1-r^2}{1-2r \cos \Theta + r^2}\). This is denoted the Poisson kernel. We write

\[ P_r(\Theta) = \frac{1-r^2}{2\pi (1-2r \cos \Theta + r^2)} \]

a) Prove directly that

\[ \lim_{r \to 1} P_r = 0 \]

is true if the limit is interpreted \(\lambda\)-a.e. or \(\lambda\)-almost uniformly, or in \(\lambda\)-measure.

b) With help of the theorems in the book, one of the above statements implies the other two. Which?
2c) Prove that
\[ \lim_{r \to 1} P_r = 0 \]
is false with respect to convergence in \( \lambda \)-mean. What should the limit be?

2d) Show that
\[ \int_{-\pi}^{\pi} P_r(\theta) \, d\lambda(\theta) = 1 \]
for all \( r \).

2e) If \( f \in C((-\pi, \pi)) \) (continuous functions on \([-\pi, \pi])\), show that
\[ \lim_{r \to 1} \int P_r(\theta) \, d\lambda(\theta) = f(0) = \int f \, d\delta_0 \]
(see 2g for definition of \( \delta_0 \)).

2f) Show that
\[ \nu_r(E) = \int_E P_r(\theta) \, d\lambda(\theta) \]
defines a Borel measure on \([-\pi, \pi])
29) Let $\delta_0$ be the Dirac measure on $[-\pi, \pi]$, i.e. $\delta_0(E) = \begin{cases} 1 & 0 \in E \\ 0 & 0 \notin E \end{cases}$

(The $\sigma$-algebra is understood to be $\mathcal{B}(\mathbb{R})$ restricted to $[-\pi, \pi]$)

Prove that

$$\lim_{r \to 1} P_r = \infty$$

$\delta_0$-a.e., $\delta_0$-almost uniformly & $\delta_0$ in mean.

In the continuation course we shall prove the fundamental Riesz-representation theorem, which here implies that "the dual" (i.e. set of all bounded linear functionals)

of $C([-\pi, \pi])$ "equals" all finite "signed" Borel measures.

The answer to the question in 2c is that the limit doesn't exist. However, thinking of $P_r$ as measures as in 2f, 2e states that

$$\lim_{r \to 1} P_r = \delta_0$$ in the weak star topology.

In other words, $P_r$ is an "approximate identity."
1. Set $X = \mathbb{N}$, $A = \mathcal{P}(\mathbb{N})$ and $\mu$ a counting measure. A sequence $(a_k)_{k=1}^\infty$ then defines a function $a(k) = a_k$ on $\mathbb{N}$. If $a_k \geq 0 \forall k$, MCT gives

$$\int (a(k) \, d\mu) = \lim_{K \to \infty} \int a(k) \chi_{\{1, \ldots, K\}}(k) \, d\mu$$

$$= \lim_{K \to \infty} \sum_{k=1}^K a_k = \sum_{k=1}^\infty a_k.$$

If $a_k \in \mathbb{C}, \forall k$, we have by Prop 2.6.4 that $\int a(k) \, d\mu$ exists iff

$$\int |a(k)| \, d\mu < \infty.$$

In this case, the DCT (with $g(k) = 1_{a_k}$) gives

$$\int (a(k) \chi_{\{1, \ldots, K\}}(k) \, d\mu = \lim_{K \to \infty} \sum_{k=1}^K a_k = \sum_{k=1}^\infty a_k.$$

Similarly, if $(n_k)_{k=1}^\infty$ is an enumeration of $\mathbb{N}$, then $a(k) \chi_{\{n_1, \ldots, n_k\}}(k)$ pointwise to $a$ as $K \to \infty$ so DCT gives

$$\lim_{K \to \infty} \sum_{k=1}^K a_k = \lim_{K \to \infty} \int a \chi_{\{n_1, \ldots, n_K\}} \, d\mu = \int a \, d\mu.$$

QED
2. As in the previous exercise we can interpret \( \sum_{k=-\infty}^{\infty} r^{|k|} e^{i k \theta} \) as integrals on \( \mathbb{Z} \). We have

\[
\sum_{k=-\infty}^{\infty} r^{|k|} e^{i k \theta} = 1 + \sum_{k=1}^{\infty} r^{k} e^{i k \theta} + \sum_{k=-1}^{-\infty} r^{-k} e^{-i k \theta}
\]

\[
= 1 + 2 \text{Re} \lim_{K \to \infty} \sum_{k=1}^{K} (re^{i \theta})^k
\]

\[
= 1 + 2 \text{Re} \lim_{K \to \infty} \frac{re^{i \theta} - (re^{i \theta})^K}{1 - re^{i \theta}} = 1 + 2 \text{Re} \frac{re^{i \theta}}{1 - re^{i \theta}}
\]

\[
= \text{Re} \frac{1 + re^{i \theta}}{1 - re^{i \theta}} - \text{Re} \frac{(1-re^{-i \theta})(1+re^{i \theta})}{|1-re^{i \theta}|^2}
\]

\[
= \text{Re} \frac{1-\frac{r^2}{1-re^{i \theta}}}{|1-re^{i \theta}|^2} = \frac{1-r^2}{|1-re^{i \theta}|^2}
\]

\[
= \frac{1-r^2}{1-2r \cos \theta + r^2}.
\]

Note that \( P_r(d) = \frac{1-r^2}{(1-r)^2} = \frac{1+r}{1-r} \).
2a) Note that $P_r$ is even & decreasing for $\Theta \in [0, \pi]$. For fix $\Theta \neq 0$ we have

$$\lim_{r \to 1} |1 - r e^{i\Theta}|^2 = |1 - e^{i\Theta}|^2 = 0,$$

so

$$\lim_{r \to 1} P_r(\Theta) = \frac{\lim_{r \to 1} 1 - r^2}{\lim_{r \to 1} |1 - r e^{i\Theta}|^2} = 0.$$ 

Thus $P_r$ converges pointwise on $([-\pi, 0) \cup (0, \pi])$, and hence $\lambda$-a.e.

Moreover, given any $\varepsilon \& \delta > 0$ we have

$$\{\Theta: P_r(\Theta) > \varepsilon^2\} \subset [-\delta, \delta]$$

if $r$ is large enough that $P_r(\Theta) \leq \varepsilon$

Thus $P_r \to 0$ uniformly on $([-\pi - \delta) \cup (\delta, \pi])$

and hence $P_r \to 0$ almost uniformly.

Also $\limsup_{r \to 1} \lambda(\Theta: |P_r(\Theta) - 0| > \varepsilon^2) \leq \lambda([-\delta, \delta]) = 2\delta$, so $\limsup_{r \to 1} \lambda(\cdot) = 0$

and hence $P_r \to 0$ in measure.
2b Since \( \lambda \) is finite on \([\pi, \pi]\), the \( \lambda \)-a.e. convergence implies the other two by Prop

2d) Fix \( r \) and set \( f_k(\theta) = \sum_{j=-k}^{k} r^{|j|} e^{ij\theta} \)

Then \( \|f_k(\theta)\| \leq \sum_{j=-k}^{k} r^{|j|} = \frac{1+r}{1-r} \). The constant function \( g(\theta) = \frac{1+r}{1-r} \) is integrable on \( X = [\pi, \pi] \), so DCT gives

\[
2\pi \int_{\pi}^{\pi} |P(\theta)| \, d\theta = \int_{\pi}^{\pi} \lim_{k \to \infty} f_k(\theta) \, d\theta = \lim_{k \to \infty} \int_{\pi}^{\pi} f_k(\theta) \, d\theta
\]

\[
= \lim_{k \to \infty} \sum_{j=-k}^{k} \int_{\pi}^{\pi} e^{ij\theta} \, d\theta = \lim_{k \to \infty} 2\pi = 2\pi
\]

2c) By the above we have

\[
\int_{\pi}^{\pi} \left| P(\theta) - O \right| \, d\theta = \int_{\pi}^{\pi} P(\theta) \, d\theta = 1
\]

so \( P \to O \) in mean.
2e With $\varepsilon, \delta$ as in 2a) we have

$$\int f P_\varepsilon \, d\lambda = \int (f - f(0)) P_\varepsilon \, d\lambda + \int f(0) P_\varepsilon \, d\lambda.$$  

Exp)

Since $\int f(0) P_\varepsilon \, d\lambda = f(0)$ and

$$\left| \int f - f(0) P_\varepsilon \, d\lambda \right| \leq \int |f - f(0)| P_\varepsilon \, d\lambda,$$

$$\leq \int |f - f(0)| P_\varepsilon \, d\lambda + \int |f - f(0)| P_\varepsilon \, d\lambda,$$

$$\leq 2 \| f \|_\infty \int \varepsilon \, d\lambda + \sup_{|0| < \delta} |f(0) - f(0)|,$$

$$\leq 2 \| f \|_\infty \cdot 2\pi \varepsilon + \sup_{|0| < \delta} |f(0) - f(0)|,$$

$$\leq 2 \| f \|_\infty \cdot 2\pi \varepsilon + \sup_{|0| < \delta} |f(0) - f(0)|,$$

Since $f$ is continuous, this can be made arbitrarily small by choosing $\delta$ small, so

$$\lim_{\varepsilon \to 0} \int f P_\varepsilon \, d\lambda = f(0),$$

as desired.