Department of
AUTOMATIC CONTROL

Exam in Optimization for Learning

2019-10-28

Points and grading

All answers must include a clear motivation. Answers should be given in English. The total number of points is 20. The maximum number of points is specified for each subproblem. Preliminary grading scales:

Grade 3: 12 points on the exam
     4: 17 points on exam plus extra-credit handin
     5: 22 points on exam plus extra-credit handin

Accepted aid

Authorized *Cheat Sheet.*

Results

Solutions will be posted on the course webpage, and results will be registered in LADOK. Date and location for display of corrected exams will be posted on the course webpage.

1. Which of the following sets S are convex?

    a. $S = \{(x, y) \in \mathbb{R}^2 : x^2 = y\}$ (1 p)

    b. $S = \{x \in \mathbb{R}^n : \max_i(x_i) \le r\}$, where $r > 0$. (1 p)

    c. $S = \{(x, t) \in \mathbb{R}^2 : |x|^2 \le t^2\}$. (1 p)

    d. $S = \text{epi}(f)$ where $f(x) = e^x$ and $x \in \mathbb{R}$. (1 p)

    e. $S = \{x \in \mathbb{R}^n : Ax \ge b\}$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. (1 p)

2. Which of the following functions f are convex? Prove or disprove.

    a. $f(x) = \begin{cases} x & \text{if } x > 0 \\ -1 & \text{if } x \le 0 \end{cases}$ where $x \in \mathbb{R}$. (1 p)

    b. $f(x) = g^*(x)$ where $g(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0$ where $x \in \mathbb{R}$. (1 p)

    c. $f(x) = \begin{cases} -\min(\log(x), -e^{-x}) & \text{if } x > 0 \\ \infty & \text{if } x \le 0 \end{cases}$ where $x \in \mathbb{R}$. (1 p)

    d. $f(x) = |x|^3$ where $x \in \mathbb{R}$. (1 p)

    e. $f(x) = \begin{cases} \|x\|_2^2 & \text{if } Ax = b \\ \infty & \text{otherwise} \end{cases}$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. (1 p)

3. Compute the proximal operator $\text{prox}_{\gamma f}(x)$ for the Huber loss

    $$f(x) = \begin{cases} \dfrac{1}{2} x^2 & \text{if } |x| \le 1/2 \\ \dfrac{1}{2}\left(|x| - \dfrac{1}{4}\right) & \text{if } |x| > 1/2 \end{cases}$$

    where $f : \mathbb{R} \to \mathbb{R}$. (1 p)

4. Compute the subdifferential to the berHu (reversed Huber) loss

    $$f(x) = \begin{cases} \|x\|_2 & \text{if } \|x\|_2 \le 1 \\ \dfrac{1}{2}(\|x\|_2^2 + 1) & \text{if } \|x\|_2 > 1 \end{cases}$$
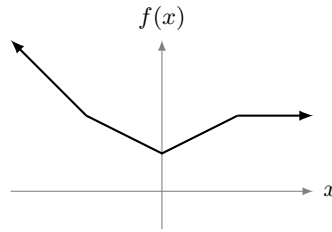
    (1 p)

5. A convex function f has the following properties: $f(-2) = 3$, $\partial f(-1) = \{-1\}$, $\partial f(0) = [-1, 0]$. What can you conclude about the following properties?

    a. Smoothness (1 p)

    b. Strong convexity (1 p)

6. Sketch the conjugate of the piecewise linear function showed below. Outside the plotted domain, assume the graph continues in the same direction as on the boundary. (1 p)

$f(x)$

x

7. Consider the functions $g(x) = \gamma f(x)$ and $h(x) = f(x - b)$ where $\gamma > 0$.

  a. Find g^* expressed in f^* and γ. (1 p)

  b. Find h^* expressed in f^* and b. (1 p)

8. Consider the coordinate proximal-gradient method (CPG)

$$\text{Choose } i$$
$$x_i^{k+1} = \text{prox}_{\gamma_i g_i}(x_i^k - \gamma_i(\nabla f(x^k))_i)$$
$$x_j^{k+1} = x_j^k \quad \forall j \neq i.$$

Let $x^\star$ be a fixed point of CPG, i.e. $x^k = x^\star \implies x^{k+1} = x^\star$, regardless of which coordinate i was chosen. Show that $x^\star$ solves the problem

$$\min f(x) + \sum_{i=1}^n g_i(x_i)$$

given that f and g_i are closed convex and constraint qualification hold. (1 p)

9. Consider primal problem

$$\min_{x \in \mathbb{R}^n} \|x\|_1 + |\mathbf{1}^T x|$$

and dual problem

$$\min_{y \in \mathbb{R}} \iota_{[-\mathbf{1},\mathbf{1}]}(-\mathbf{1}y) + \iota_{[-1,1]}(y)$$

where $\mathbf{1} \in \mathbb{R}^n$ is a vector of all ones. Then function $\iota_{[a,b]}$ is the indicator of the set $\{x \in \mathbb{R}^m : a \leq x \leq b\}$ where the inequality is applied element wise.

Let $y^\star$ be a solution to the dual problem and assume the primal solution $x^\star$ is unknown. Recover $x^\star$ from $y^\star$ using the primal-dual optimality conditions. (2 p)