1) The following figures show different realizations, covariance functions and spectral densities. The processes are either autoregressive (AR(p)) or moving average (MA(q)). Determine, with a motivation, what figures that are realizations, covariance functions and spectral densities. Also state which realization, covariance function and spectral density that are connected. Decide and motivate of which type the different processes are (AR or MA) and which orders, $p$ or $q$ they have. The orders can be assumed to be smaller than 10.

(10p)

2) The zero-mean process $X(t), t \in \mathbb{R}$, is disturbed by the zero-mean noise process $N(t), t \in \mathbb{R}$. The covariance functions are,

$$r_X(\tau) = e^{-|\tau|}$$

and

$$r_N(\tau) = \alpha e^{-\beta|\tau|}.$$

To reconstruct the process $X(t)$, a filter should be used that minimizes $E[(Y(t) - X(t))^2]$ where $Y(t) = X(t) + N(t)$, (Wiener filter). Determine the frequency function of the filter.

(10p)
3) A real-valued continuous-time process $X(t)$, $t \in \mathbb{R}$, is described by the spectral density according to the figure below. The process is sampled with the sampling distance $d = 1/40$. Determine the spectral density for the sampled process? The answer should be motivated by calculations or figures.

4) An accelerometer can measure, e.g., vibrations in the ground. We want to use a number of accelerometers to measure the vibrations from a train passing a housing area. For a reliable value we can average the $n$ measurements from a number of different accelerometers, $X_i$, which are located at a certain distance $d$ from each other on a line,

$$\hat{m}_n = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

where $X_i$, $i = 1...n$, is defined as stochastic variables with variance $V[X_i] = 1$. The covariance between two accelerometers of distance $d$ is

$$r(d) = e^{-d/10}.$$

The accelerometers cost 3,000 Skr each and the cost of the cable between the accelerometers is 100 Skr/m. Assume that your budget is maximized to 12,000 Skr to be spent on accelerometers and cable. The example below shows how a number of 3 accelerometers is connected, (series connection).

With the restrictions described above, how many accelerometers should you use for the average $\hat{m}_n$ to have as low variance as possible and what value do you get for the variance? Hint: One can show that the smallest variance is given when measurements are placed at equidistant positions. This is assumed to be known.
The following exercise is a simplified illustration to the problem of separating a slow and fast variation of a time series using a moving average. Let \( X_t, t = 0, \pm 1, \pm 2 \ldots \) be a weakly stationary process with covariance function

\[
r_X(\tau) = \begin{cases} 
2^{-|\tau|} + 1 & \text{for } \tau = 0 \\
2^{-|\tau|} - 1/2 & \text{for } \tau = \pm 1 \\
2^{-|\tau|} + 1/6 & \text{for } \tau = \pm 2 \\
2^{-|\tau|} & \text{for } |\tau| \geq 3
\end{cases}
\]

This means that \( X_t \) is a sum of a process with the covariance function \( 2^{-|\tau|} \) and an MA(3)-process.

Here are two alternatives to separate them: For every \( t \) the averages

\[
Z_t = \frac{X_{t+1} + X_t + X_{t-1}}{3},
Y_t = \frac{X_t + X_{t-1} + X_{t-2}}{3},
\]

are calculated. The differences \( U_t = X_t - Z_t \) and \( V_t = X_t - Y_t \) are also computed. The processes \( Z_t \) and \( Y_t \) are two possible estimates to represent the slow variation (low-pass filtering) of the process \( X_t \). Similarly, the two resulting processes \( U_t \) and \( V_t \) represent the fast variation (high-pass filtering) of the process \( X_t \).

a) Compute and compare the covariance functions \( r_Z(\tau) \) and \( r_Y(\tau) \).

b) Calculate the cross-covariance functions \( r_{X,U}(\tau) \) and \( r_{X,V}(\tau) \) and the corresponding cross-spectral densities \( R_{X,U}(f) \) and \( R_{X,V}(f) \). Hint: It is convenient to use frequency functions.

c) Use the results in (b) to explain why it is better to use the formulation with \( Z_t \) and \( U_t \) than using \( Y_t \) and \( V_t \) to separate the process \( X_t \) in the slow variations and the faster variations.

6) Determine the spectral density \( R_Y(f) \), the covariance function \( r_Y(\tau) \) and the variance \( V[Y(t)] \) if

\[
Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u) \, du
\]

when \( X(t), t \in \mathbb{R} \), is a continuous-time stationary process with expected value zero and covariance function

\[
r_X(\tau) = 1/(1 + \tau^2), \quad -\infty < \tau < \infty.
\]

The impulse response of the filter is

\[
h(t) = 1/(4 + t^4), \quad -\infty < t < \infty.
\]