Example 1. Simple strong Lyapunov function.

> Exercise 15 Show that $(x(t), y(t)) = (0,0)$ is an asymptotically stable solution of
>
> $$\begin{cases} \dot{x} = -x^3 + 2y^3 \\ \dot{y} = -2xy^2. \end{cases}$$

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Example 2. Stability by Linearization

For the following system of equations find all equilibrium points and investigate their stability and their type by linearization.

$$\begin{cases} x' = & \ln(2 - y^2) \\ y' = & \exp(x) - \exp(y) \end{cases}$$

1. **Solution.** There are two equilibrium points: $x_1 = (1,1)$and $x_2 = (-1,-1)$.

The Jacobian of the right hand side is: $\begin{bmatrix} 0 & -2\frac{y}{-y^2+2} \\ e^x & -e^y \end{bmatrix}$. Its values in x_1 and x_2 are $A_1 = \begin{bmatrix} 0 & -2 \\ e & -e \end{bmatrix}$, and $A_2 = \begin{bmatrix} 0 & 2 \\ 1/e & -1/e \end{bmatrix}$. The eigenvalues to A_1 are $-\frac{1}{2}e - \frac{1}{2}\sqrt{e^2 - 8e}$,and $\frac{1}{2}\sqrt{e^2 - 8e} - \frac{1}{2}e$ that are conjugate complex numbers with negative real parts. Therefore we observe stable spiral around the equilibrium point x_1. The eigenvalues to A_2 are , eigenvalues: $\frac{1}{e}\left(-\frac{1}{2}\sqrt{8e+1} - \frac{1}{2}\right), \frac{1}{e}\left(\frac{1}{2}\sqrt{8e+1} - \frac{1}{2}\right)$, one postive and one negative. Therefore we x_2 is a saddle point and is unstable.

Example 3.

Consider the following system of ODEs: $\begin{cases} x' = & 2y \\ y' = & -x - (1 - x^2)y \end{cases}$.

Show the asymptotic stability of the equilibrium point in the origin and find it's domain of attraction.

Solution.

We try the test function $V(x, y) = x^2 + Ay^2$ that leads to cancellation of mixed terms in the directional derivative V_f along trajectories:

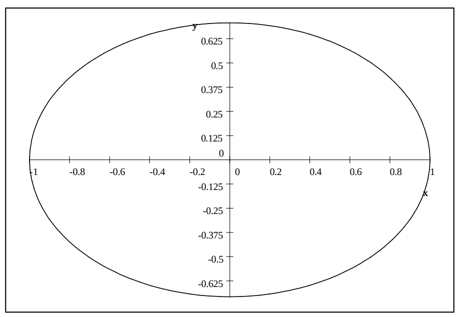
$V_f(x,y) = \nabla V \cdot f(x) = 2x2y + \left(2Ay\left(-x-(1-x^2)y\right)\right) = 4xy - 2Axy - 2Ay^2(1-x^2)$

Choose $A=2$ to cancel indefinite terms.$V(x,y) = x^2+2y^2$

$V_f(x,y) = 4xy - 4xy - 4y^2(1-x^2) = -4y^2(1-x^2)$ that is not positive for $|x| \leq 1$. Therefore the origin is a stable stationary point.

Checking the behavior of the system on the set of zeroes to $V_f(x,y)$ inside the stripe $|x|<1$ we consider $(V_f)^{-1}(0) = \{(x,y) : y=0, |x|<1\}$. On this set $y' = -x$ and the only invariant set in $(V_f)^{-1}(0)$ is the origin. The LaSalle invariance principle implies that the origin is asymptotically stable.

The domain of attraction is the largest set bounded by a level set of $V(x,y) = x^2+2y^2$ inside the stripe $|x| \leq 1$. The largest such set will be the interior of the ellipse $x^2+2y^2 = C$ such that is touches the lines $x = \pm 1$. Taking points $(\pm 1, 0)$ we conclude that $1 = C$ and the boundary of the region (domain) of attraction is the ellipse $x^2+2y^2=1$ with halfs of axes 1 and $\sqrt{0.5}$:



How to find a Lyapunov function?

If the right hand side of the equation is a higher degree polynomial, then it is often convenient to find a Lyapunov's function in a systematic way in the form of polynomial with unknown coefficients and unknown even degrees like $2m$.

Consider the system

$$\begin{aligned} x' &= -3x^3 - y \\ y' &= x^5 - 2y^3 \end{aligned}$$

Try a test function $V(x,y) = ax^{2m} + by^{2n}$, $a, b > 0$.

$$\begin{aligned} V_f(x,y) &= \nabla V \cdot f(x,y) = \\ &= a2m(x)^{2m-1} \cdot \left(-3x^3 - y\right) + b2n(y)^{2n-1}\left(x^5 - 2y^3\right) \\ &= \underbrace{-6amx^{2m+2}}_{good<0} \underbrace{-2ma(x)^{2m-1}y}_{bad-indefinite} + \underbrace{2nby^{2n-1}x^5}_{bad-indefinite} \underbrace{-4nby^{2n+2}}_{good<0} \end{aligned}$$

We choose first powers m and n so that indefinit terms would have same powers of x and y.

$$\begin{aligned} 2m - 1 &= 5; \Longrightarrow m = 3 \\ 2n - 1 &= 1; \Longrightarrow n = 1 \end{aligned}$$

Then $V_f(x,y) = -18ax^8 - 6x^5y + 2bx^5y - 4nby^4$.We choose $a = 1$ and $b = 3$ to cancel indefinite terms. Then

$$\begin{aligned} V(x,y) &= x^6 + 3y^2 \\ V_f(x,y) &= -18x^8 - 12y^4 < 0, \quad (x,y) \neq (0,0) \end{aligned}$$

Therefore V is a strong Lyapunov's function in the whole plane and the equilibrium is a globally asymptotically stable equilibrium point, because $V(x,y) = x^6 + 3y^2 \to \infty$ as $\|(x,y)\| \to \infty$.

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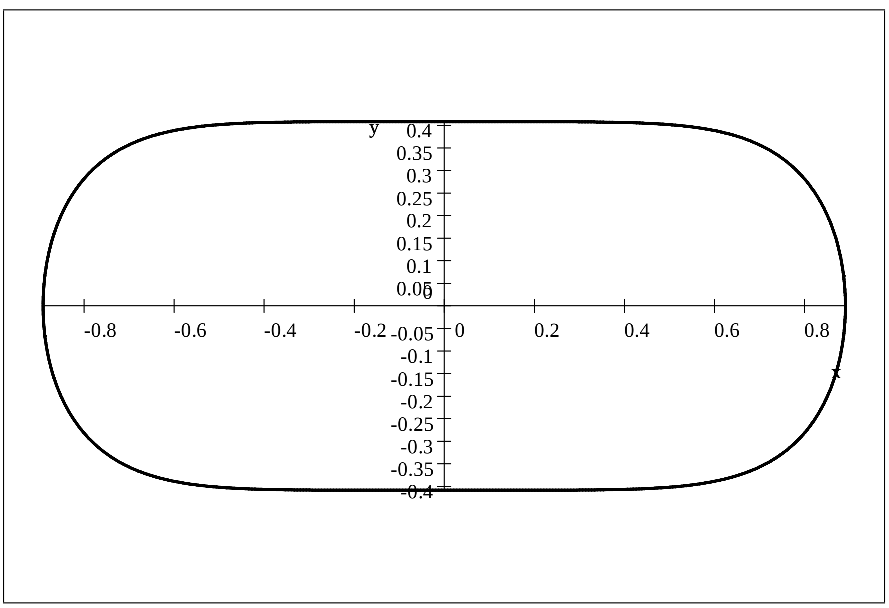
Example 4. Investigate stability of the equilibrium point in the origin.

$$x' = -y - x^3$$
$$y' = x^5$$

We try our simplest choice of the Lyapunov function: $V(x, y) = x^2 + y^2$ and arrive to

$$V_f(x, y) = -2xy - 2x^4 + 2yx^5$$

It does not work because the expression $V_f(x, y)$ includes two indefinite terms: $2xy$ and $2yx^5$ that change sign around the origin. We try a more flexible expression by looking on particular expressions in the right hand side of the equation: $V(x, y) = x^6 + \alpha y^2$ where $\partial V/\partial x = 6x^5$ with the same power of x as in the equation, and the parameter α that can be adjusted later. V is a positive definite function: $V(0) = 0$ and $V(z) > 0$ for $z \neq 0$.The level sets to V look as flattened in y - direction ellipses. The curve $x^6 + 3y^2 = 0.5$ is depicted:



$$V_f(x, y) = 6x^5(-y - x^3) + 2\alpha yx^5 = -6x^5y + 2\alpha x^5y - 6x^8$$

We get again two indefinite terms, but they are proportional and the choice

$\alpha = 3$ cancels them:

$$V_f(x, y) = -6x^8 \leq 0$$

Therefore the origin is a stable equilibrium point. $V_f(x, y) = 0$ on the whole $y-$axis that in our "general" theory is denoted by $V_f^{-1}(0)$.We check invariant sets of the system on the set $V_f^{-1}(0)$. We observe that $x' = -x^3$ (only this fact is important) and $y' = 0$ (it does not matter for $V_f^{-1}(0)$ that is $y-$axis). Therefore $\{0\}$ is the only invariant set on the y - axis. Trajectories starting on the y - axis go across it in all points except $\{0\}$. The LaSalle's invariance principle implies that all trajectories approach $\{0\}$ as t tends to infinity and the origin is asymptotically stable.

The test function $V(z) \to \infty$ as $\|z\| \to \infty$. It implies that the whole plain is a region or domain of attraction for the equilibrium point in the origin.

How to find a strong Lyapunov's function?

Example 4.

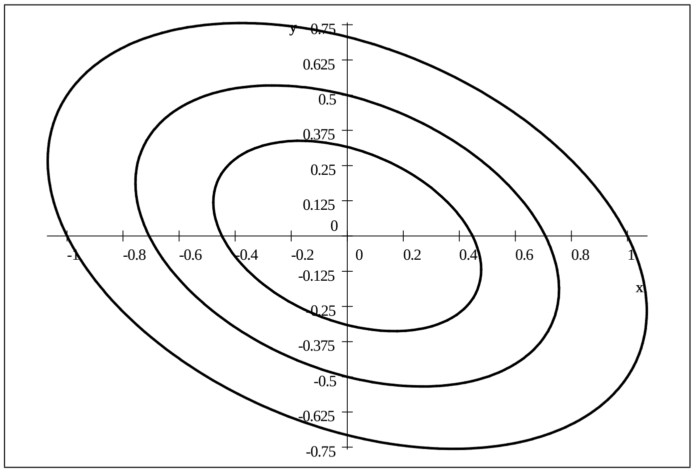
It is theoretically possible to find a strong Lyapunov function for the same system as in the Example 3.

Looking on the previous week Lyapunovs function $x^6 + 3y^2$ we see that it's "weekness" followed from the fact that both level sets of V and velocities of the system were orthogonal to the y - axis. It implied that $V_f(z) = 0$ on the y - axis. To go around this problem a strong Lyapunov function must have level sets that deviate slightly from the normal to the y - axis. Adding a relatively small indefinite term xy^3 to the function $x^6 + 3y^2$ we get this effect. A level set corresponding $x^6 + xy^3 + 3y^2 = 0.7$ of this new Lyapunovs function looks as a slightly rotated version of level sets for the previous (weak) Lyapunovs function.

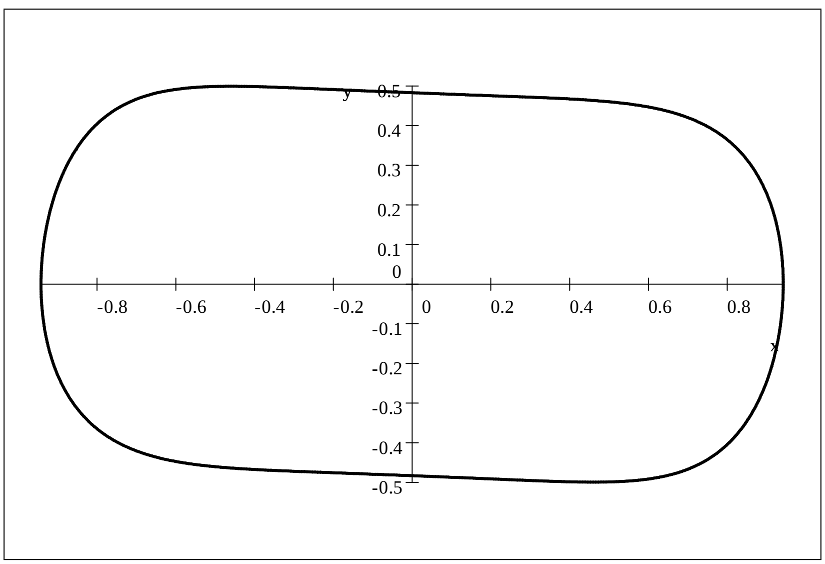
Why like that ? Take a simpler example with an ellipse curve $x^2 + 2y^2 = 1$ and another that is $x^2 + xy + 2y^2 = 1$

This quadratic form is positive definite: the matrix is $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}$.A quadratic form $\mathbf{x}^T A\mathbf{x} = Q(\mathbf{x})$ with $\mathbf{x} = [x, y]^T$ is positive definite if and only if $\det A > 0$ and all submatrices A_i from the upper left corner have positive determinants: $\det A_i > 0$.

Level sets of the positive definite quadratic form with mixed tems like $x^2 + xy + 2y^2$ are ellipses with symmetry axes (that are orthogonal eigenvectors to A) and are rotated with respect to coordinate axes:



We try to introduce the test function $V(x,y) = x^6 + xy^3 + 3y^2$ with an indefinite mixed term xy^3 added, that would similarly with the ellipses, give slightly rotated level sets so that trajectories would cross them strictly inside on the y - axis:



We claim that the test function $V(x,y) = x^6 + xy^3 + 3y^2$ is positive definite and is a strong Lyapunovs function namely that $V_f(x,y) < 0$ for $(x,y) \neq (0,0)$.

Because of the geometry of the vector field f of our equation $z' = f(z)$ velocities on the y axis cross such level sets strictly towards inside, implying the desired strict inequality $V_f(z) < 0$, $z \neq 0$ on the y axis. We need to check that

$V(x,y) = x^6 + xy^3 + 3y^2$ is positive definite (it is not trivial) and to show that $V_f(z) < 0$, $z \neq 0$ for all $z \in \mathbb{R}^2$ (it requires some non-trivial analysis).

A very useful inequality in analysis is

Young's inequality

Lemma. If $a, b \geq 0$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for every pair of numbers $p, q \in (1, \infty)$ satisfying the conjugacy relation.

$$\frac{1}{p} + \frac{1}{q} = 1$$

The simplest example of Young's inequality:

$$ab \leq \frac{1}{2}\left(a^2 + y^2\right)$$

We show that the test function $V(x,y) = x^6 + xy^3 + 3y^2$ is positive definite in a domain around the origin.

Now, let $V = x^6 + xy^3 + 3y^2$. Applying Young's inequality with $a = |x|$, $b = |y|^3$, $p = 6$, and $q = 6/5$, we see that

$$|xy^3| = |x||y|^3 \leq \frac{|x|^6}{6} + \frac{5|y|^{18/5}}{6} \leq \frac{1}{6}x^6 + \frac{5}{6}y^2$$

if $|y| \leq 1$, so

$$V \geq \frac{5}{6}x^6 + \frac{13}{6}y^2$$

if $|y| \leq 1$. Also,

We calculate $V_f = \dot{V}$ for the system from the Example 3:

$$\begin{aligned} x' &= -y - x^3 \\ y' &= x^5 \end{aligned}$$

$$\dot{V} = -6x^8 + y^3\dot{x} + 3xy^2\dot{y} = -6x^8 - y^3(y + x^3) + 3x^6y^2$$
$$= -6x^8 - x^3y^3 + 3x^6y^2 - y^4.$$

Applying Young's inequality to the two mixed terms in this orbital derivative, we have

$$|-x^3y^3| = |x|^3|y|^3 \le \frac{3|x|^8}{8} + \frac{5|y|^{24/5}}{8} \le \frac{3}{8}x^8 + \frac{5}{8}y^4$$

if $|y| \le 1$, and

$$|3x^6y^2| = 3|x|^6|y|^2 \le 3\left[\frac{3|x|^8}{4} + \frac{|y|^8}{4}\right] = \frac{9}{4}x^8 + \frac{3}{4}y^8 \le \frac{9}{4}x^8 + \frac{3}{64}y^4$$

if $|y| \le 1/2$. Thus,

$$\dot{V} \le -\frac{27}{8}x^8 - \frac{21}{64}y^4$$

if $|y| \le 1/2$, so, in a neighborhood of 0, V is positive definite and $\dot{V}$ is negative definite, which implies that 0 is asymptotically stable.

Example 5.

Consider the Lienard equation: $x'' + x' + g(x) = 0$, and investigate stability of the equilibrium in the origin. The second order equation can be rewritten as a system $z' = f(z)$:

$$\begin{aligned} x' &= y \\ y' &= -g(x) - y \end{aligned}$$

where g satisfies the following hypothesis: g is continuously differentialble for $|x| < k$ for some $k > 0$, $xg(x) > 0$, $x \neq 0$.

Solution.

Physically this equation is a Newton equation for a non-linear spring. For example if $g(x) = \sin(x)$ it describes a pendulum with friction where air resistance is proportional to velocity.

A Lyapunov function is naturally to choose as a total energy of the system:

$$V(x, y) = \frac{(y)^2}{2} + \int_0^x g(s)ds$$

Indeed it is positive definite in the region $\Omega = \{(x, y) : |x| < k \quad \}$ because $g(s)s > 0$ in Ω according to given conditions. The directional derivative of V along f is

$$V_f(x, y) = y(-g(x) - y) + g(x)y = -(y)^2$$

V is a Lyapunov's function, but not strong because $V_f(x, y) = 0$ on the whole x - axis. Therefore $V_f^{-1}(0)$ is the whole x - axis. Checking values of f on $V_f^{-1}(0)$ we observe that trajectories of the system are orthogonal to $V_f^{-1}(0)$ in all points on $V_f^{-1}(0)$ except the origin. It implies that $\{0\}$ is the only invariant set on $V_f^{-1}(0)$ that attracts all trajectorie starting in a small neighborhood of the origin. Therefore the origin is asymptotically stable.

Our next problem is to find a possibly large domain or region of attraction for the equilibrium point.If we find a closed level set for V in Ω, it will be a boundary for a domain of attraction. It will might not be the largest possible and depends on a clever choice of Lyapunov's function V.

We cannot solve this problem for a general expression $V(x, y) = \frac{(y)^2}{2} + \int_0^x g(s)ds$.

Conclusion

The lesson from the last example is that if you have got an expression for $V_f(x, y)$ like

$$V_f(x, y) = -x^2 + \frac{3}{2}xy - y^2 \leq 0$$

where ypou cannot directly state if it is always negative or not, apply the

Young's inequality

to estimate $|x|\,|y|$ in terms of x^2 and y^2.

Example 6.

Find all equilibriums, investigate their stability properties and find possible regions of attraction.

Choose a particular $g(x) = x + x^2$ in the previous example.

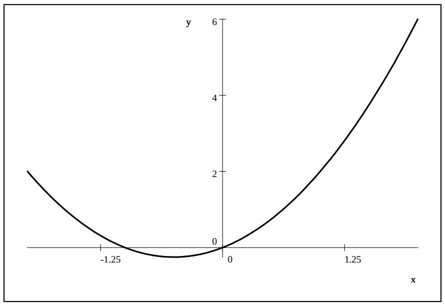
$$\begin{aligned} x' &= y \\ y' &= -(x + x^2) - y \end{aligned}$$

Observe that the system has two equilibrium points: $(-1, 0)$ and $(0, 0)$

Linearization gives Jacoby matrix $A(x, y) = \begin{bmatrix} 0 & 1 \\ -1 - 2x & -1 \end{bmatrix}$; $A(-1, 0) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ Observe that $\det \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = 0 - 1 = -1 < 0$ it implies by the Grobman - Hartman theorem, that $(-1, 0)$ is a saddle point.

$A(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, $\det \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = 1 > 0$, $trace \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = -1 < 0$,

$(trace A(0,0))^2 / 4 = 1/4 < 1 = \det A(0, 0)$. It imples that the origin is an asymptotically stable focus for the linearized system and is asymptotically stable for the original system.



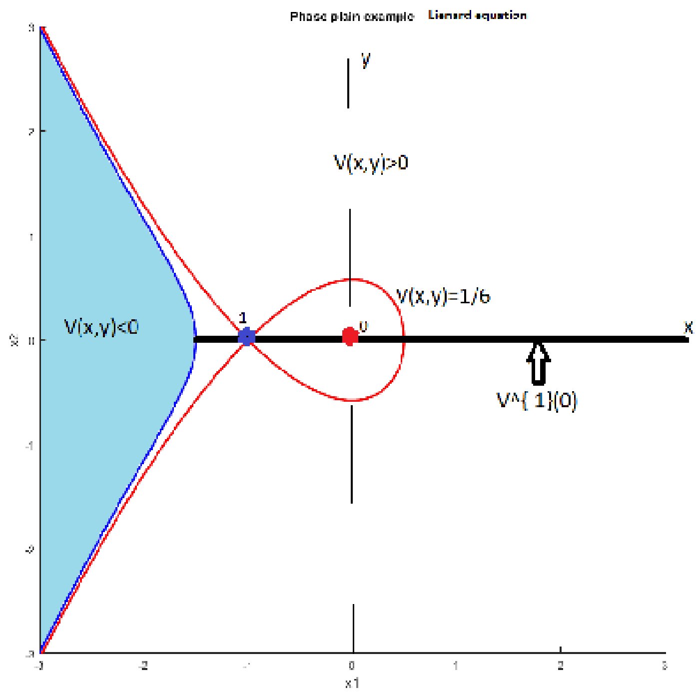
$g(x) = x + x^2$

We can find an explicit expression for the Lyapunov's function $V(x, y) = \frac{(y)^2}{2} + \int_0^x g(s)ds$.

$$V(x, y) = \frac{(x)^2}{2} + \frac{(x)^3}{3} + \frac{(y)^2}{2}$$

This function is positive definite on the set $\Omega = \left\{ (y)^2 > -(x)^2 - \frac{2}{3}(x)^3 \right\}$

The level set $\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3 = \frac{1}{6}$ is depicted by the red line.The level set $\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3 = 0$ is depicted by the blue line. We will investigate them analytically a bit later.



$V_f(x,y) = \nabla V(x,y) \cdot f = xy + (x)^2 y - (y)^2 - xy - (x)^2 y = -(y)^2 \leq 0$ valid in the whole plane $\mathbb{R}^2$.

We check which invariant sets are contained in $V_f^{-1}(0)$ on Ω that is a part of x - axis $\{(x,0) : x > -3/2\}$ that is a thick black line on the picture above.

Notice that $V_f^{-1}(0)$ on Ω contains two equilibrium points $(-1,0)$ and $(0,0)$ and they both are invariant sets. We like to find a largest domain $\Omega_1 \subset \Omega$ bounded by a part of a level set of V such that Ω_1 does not include the point $(-1,0)$. Then Ω_1 contains only one invariant set that is the origin $(0,0)$. This set Ω_1 is the domain of attraction for the asymptotically stable equilibrium in $(0,0)$.

Such largest level set of V must go through the second equilibrium point $(-1,0)$ and it's value there is $V(x,y) = V(-1,0) = 1/6$. The domain of attraction Ω^* is the egg - shaped domain bounded by the closed curve $(y)^2 =$

$1/3 - \left((x)^2 + \frac{2}{3}(x)^3\right)$ or as a union of two explicit branches:

$$y = \pm\sqrt{1/3 - \left((x)^2 + \frac{2}{3}(x)^3\right)}$$

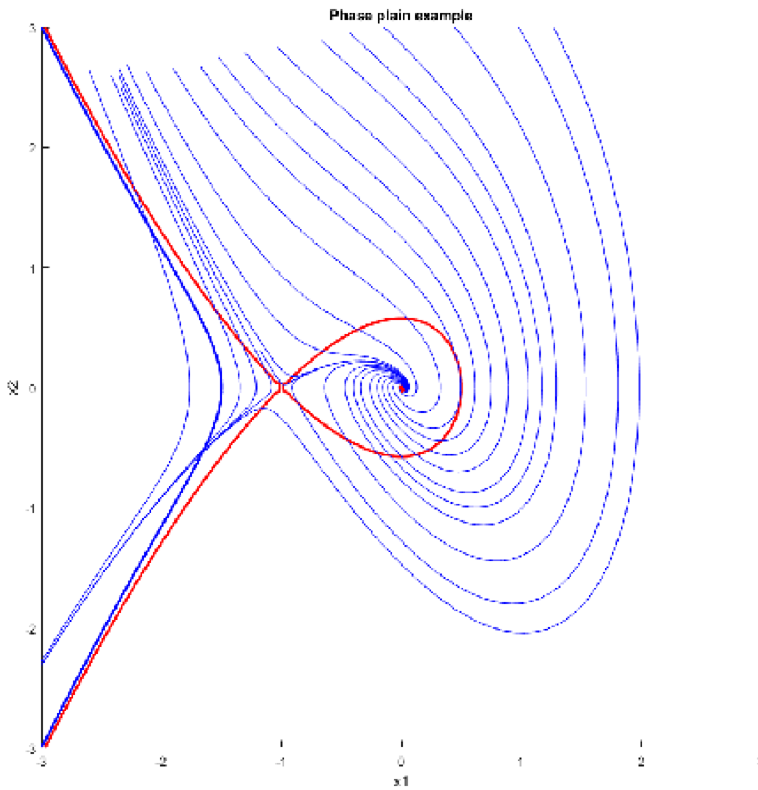
It is a part of the red level set on the picture. To see that this curve is closed we consider derivative of the function

$\frac{d}{dx}\left(1/3 - \left((x)^2 + \frac{2}{3}(x)^3\right)\right) = -2x - 2x^2 = (-2)\,x\,(x+1)$. It implies that the functions has a maximum in $x = 0$, and minimum at $x = -1$. $V(x)$ has zero in $x = -1$ and another zero in $x = 1/2$:

$1/3 - \left((x)^2 + \frac{2}{3}(x)^3\right)\Big|_{x=1/2} = 1/3 - \left((1/2)^2 + \frac{2}{3}(1/2)^3\right) = 1/3 - \left((1/4) + \frac{1}{3}(1/4)\right) = 1/3 - 1/3 = 0,$

■

One can try to find an even larger region of attraction Ω^{**} for the equilibrium point in the origin. It cannot include the equilibrium in $(-1, 0)$ because it is unstable (a saddle point). We can extend Ω_1 to a rectangle $[-1, 0] \times \left[0, \sqrt{3}/3\right]$in the second quadrant by checking signs of x' and y' on it's left and upper sides. Actual region of attraction is even a bit larger as one can see on the phase portrait



Example 7. Exercise 5.13 from L.R.

Investigate stability of the equilibrium point in the origin and find a possible domain of attraction for the following system.

$$
\begin{aligned}
x_1' &= -x_2(1 + x_1 x_2) \\
x_2' &= 2x_1
\end{aligned}
$$

We try choose the Lyapunov function V as

$$V(x_1, z_2) = 2x_1^2 + x_2^2$$

We could try first a function $V(x_1, x_2) = ax_1^2 + x_2^2$, check V_f and then decide which value a suites best.

$$
\begin{aligned}
V_f(x_1, x_2) &= \nabla V \cdot f(x_1, x_2) = -2ax_1x_2(1 + x_1x_2) + 2x_2 2x_1 \\
&= 4x_1x_2 - 2ax_1x_2 - 2ax_1^2x_2^2 = -2ax_1^2x_2^2 \leq 0 \\
for \quad a &= 2
\end{aligned}
$$

We conclude that the equilibriom 0 is stable. $V_f\ (x_1, x_2) = -2ax_1^2x_2^2 = 0$ on both coordinate axes. We check which invariant sets are contained in $V_f^{-1}\ (0)$.

If $x_1 = 0$, then $x_1' = -x_2$, $x_2' = 0$. Therefore only $\{0\}$ is an invariant set on the x_2 axis.

If $x_2 = 0$, then $x_1' = 0$, $x_2' = 2x_1$. Therefore only $\{0\}$ is an invariant set on the x_1 axis.

Trajectories $\varphi(t, \xi)$ starting inside ellipses $V(x_1, z_2) = 2x_1^2 + x_2^2 = C > 0$ are contained inside these ellipses because $\nabla V \cdot f(x) \leq 0$. It implies that their positive orbits $O_+(\xi)$ are bounded and have compact closure in $\mathbb{R}^2$.

It implies according to the LaSalle's theorem that all these solutions $\varphi(t, \xi)$ approach the maximal invariant set in $V_f^{-1}\ (0)$ that in our particular case consists of just one point $(0, 0)$. Therefore the equilibrium point in the origin is asymptotically stable. It is also globally stable because the Lyapunov function $V(x)$ tends to infinity as $\|x\| \to \infty$, making that arbitrary large elliptic discs from the family $2x_1^2 + x_2^2 < C$ are regions of attraction.

■

Example 8. This example demonstrates how to use Young inequality for estimating $V_f(x, y)$

Consider the following system of ODE: $\begin{cases} x' = & -x - 2y + xy^2 \\ y' = & 3x - 3y + y^3 \end{cases}$.

1. Show asymptotic stability of the equilibrium point in the origin and find the region of attraction for that.

   Hint: applying Lyapunovs theorem, use the elementary Young's inequality $2xy \leq \left(x^2 + y^2\right)$ to estimate indefinite terms with xy. **(4p)**

   Solution. Choose a test function $V(x, y) = \frac{1}{2}\left(x^2 + y^2\right)$

   $V_f = x(-x - 2y + xy^2) + y\left(3x - 3y + y^3\right) = xy - x^2 - 3y^2 + y^4 + x^2y^2$

   $= -x^2\left(1 - y^2\right) - y^2\left(3 - y^2\right) + xy \leq 0 \quad ?????$

We apply the inequality $|x|\,|y| \leq \frac{1}{2}\left(x^2+y^2\right)$ to the last term and collecting terms with x^2 and y^2 arrive to the estimate

$V_f \leq -x^2\left(0.5-y^2\right) - y^2\left(2.5-y^2\right)$

It implies that $V_f(x,y) < 0$ for $(x,y) \neq (0,0)$ and $|y| < 1/\sqrt{2}$.Therefore the Lyapunov function is strong and the origin is asymptotically stable.

The attracting region is bounded by the largest level set of V - a circle having the center in the origin that fits to the domain $|y| < 1/\sqrt{2}$, namely $\left(x^2+y^2\right) < 1/2$.

Another more clever choice of a test function is $V(x,y) = 3x^2 + 2y^2$.

$V_f = 6x(-x-2y+xy^2) + 4y(3x-3y+y^3) = 4y^4 - 12y^2 - 6x^2 + 6x^2y^2 =$
$-4y^2\left(3-y^2\right) - 6x^2\left(1-y^2\right) < 0$

for $|y| < 1$, therefore the ellipse $3x^2 + 2y^2 < 2$ is a domain of attraction for the asymptotically stable equilibrium in the origin.

One can also observe the asymptotic stability of the origin by linearization with variational matrix

$A = \begin{bmatrix} -1 & -2 \\ 3 & -3 \end{bmatrix}$, with characteristic polynomial: $\lambda^2 + 4\lambda + 9 = 0$, and calculating eigenvalues: $-i\sqrt{5}-2, i\sqrt{5}-2$ with $\operatorname{Re}\lambda < 0$. But the linearization gives no information about the region of attraction.■

Example 9 on instability

Consider the following system of ODEs. Prove the instability of the equilibrium point in the origin, of the following system

$$\begin{cases} x' = & x^5 + y^3 \\ y' = & x^3 - y^5 \end{cases}$$ **(4p)**

using the test function $V(x,y) = x^4 - y^4$ and Lyapunov's instability theorem.

Solution.

Denoting $f(x,y) = \begin{bmatrix} x^5+y^3 \\ x^3-y^5 \end{bmatrix}$, consider how $V(x,y) = x^4 - y^4$changes along trajectories of the system. $f(x,y)\cdot\nabla V(x,y) = \begin{bmatrix} x^5+y^3 \\ x^3-y^5 \end{bmatrix} \cdot \begin{bmatrix} 4x^3 \\ -4y^3 \end{bmatrix} =$

$x^5 4x^3 + y^3 4x^3 - x^3 4y^3 + y^5 4y^3 = x^5 4x^3 + y^5 4y^3 = 4(x^8+y^8) > 0.$

Point out that the function $V(x,y) = x^4 - y^4$ is positive along the line $y = x/2$, $x > 0$ arbitrarily close to the origin. It implies according to the instability theorem, that the origin is an unstable equilibrium.■