Big group 3rd of February: 3.3:9; 3.3:10b,c; 3.4:3; räkna själv: 3.3:1; 3.3:2

Small group 4th and 7th of February: 3.5:4; Eö 23; 4.2:1

3.3:9

Suppose that $\{\phi_n\}_1^\infty$ is an orthonormal basis for $L^2(a,b)$. Show that for any functions $f, g \in L^2(a,b)$,

$$\langle f, g\rangle = \sum_{n=1}^{\infty} \langle f, \phi_n\rangle \overline{\langle g, \phi_n\rangle}$$

Solution: We know from this chapter that functions $f, g \in L^2(a,b)$ can be written as linear combinations of basis functions, i.e., as (generalized) Fourier series:

$$f = \sum_{n=1}^{\infty} \langle f, \phi_n\rangle \phi_n, \qquad g = \sum_{m=1}^{\infty} \langle g, \phi_m\rangle \phi_m$$

Use the properties of the inner product (§3.1 in Folland):

$$\begin{aligned}
\langle f, g\rangle &= \langle f, \sum \langle g, \phi_m\rangle \phi_m\rangle && \text{(sum representation of } g) \\
&= \overline{\langle \sum \langle g, \phi_m\rangle \phi_m, f\rangle} && \langle f_1, f_2\rangle = \overline{\langle f_2, f_1\rangle} \\
&= \sum \overline{\langle g, \phi_m\rangle \langle \phi_m, f\rangle} \\
&= \sum \overline{\langle g, \phi_m\rangle} \langle f, \phi_m\rangle && \text{(sesquilinearity)}
\end{aligned}$$

3.3:10a,b,c

Evaluate the following series by applying Parseval's equation to certain of the Fourier expansions in Table 1 of §2.1.

$$\text{a.} \quad \sum_1^\infty \frac{1}{n^4} \qquad \text{b.} \quad \sum_1^\infty \frac{1}{(2n-1)^6} \qquad \text{c.} \quad \sum_1^\infty \frac{n^2}{(n^2+1)^2}$$

Solution: From the previous exercise, we deduce the following:

Theorem 0.1 (Parts of Theorem 3.4)**.** *Let $\{\phi_n\}_1^\infty$ be an orthonormal set in $L^2(a,b)$. For every $f \in L^2(a,b)$,*

$$\|f\|^2 = \sum_1^\infty |\langle f, \phi_n\rangle|^2 \quad \Longleftrightarrow \quad f = \sum_1^\infty \langle f, \phi_n\rangle \phi_n$$

Let $(a,b) = (-\pi, \pi)$. We find in Table 1 of §2.1 that

$$f(t) = t^2 = \frac{\pi^2}{3} + \sum_1^\infty \frac{4(-1)^n}{n^2} \cos(nt)$$

Recall that a basis for $L^2(-\pi, \pi)$ is

$$\{\cos nx\}_{n=0}^{\infty} \cup \{\sin nx\}_{n=1}^{\infty}$$

Writing

$$f(x) = \frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Parseval's equality takes the form

$$\|f\|^2 = \frac{1}{2}|a_0|^2 + \sum_{1}^{\infty}\left(a_n^2 + b_n^2\right)$$

We identify

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}, \quad b_n = 0$$

so

$$\|f\|^2 = \frac{1}{2}\left(\frac{2\pi^2}{3}\right)^2 + \sum_{1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)^2 = 2\frac{\pi^4}{9} + 16\sum_{1}^{\infty}\frac{1}{n^4}$$

Since

$$\|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi}x^4 dx = \frac{2}{\pi}\frac{\pi^5}{5} = 2\frac{\pi^4}{5}$$

Equating

$$2\frac{\pi^4}{5} = 2\frac{\pi^4}{9} + 16\sum_{1}^{\infty}\frac{1}{n^4}$$

we get

$$4\frac{\pi^4}{45} = 8\sum_{1}^{\infty}\frac{1}{n^4} \iff \frac{\pi^4}{90} = \sum_{1}^{\infty}\frac{1}{n^4}$$

When $f(\theta) = \theta(\pi - |\theta|)$

$$f(\theta) = \sum_{1}^{\infty}\frac{8}{\pi(2n-1)^3}\sin(2n-1)\theta$$

so $a_n = 0$ and for even n we have $b_n = 0$. For odd n,

$$b_n = \frac{8}{\pi n^3} \quad \text{i.e.} \quad b_{2n-1} = \frac{8}{\pi(2n-1)^3}.$$

$$\|f\|^2 = \sum_{n=1}^{\infty}\left|\frac{8}{\pi(2n-1)^3}\right|^2 = \frac{64}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}$$

Now

$$\|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2(\pi - |x|)^2 dx = \frac{2}{\pi}\int_{0}^{\pi}x^2(\pi - x)^2 dx = \frac{\pi^4}{15}$$

so

$$\frac{\pi^4}{15} = \frac{64}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6} \qquad \text{or} \qquad \frac{\pi^6}{960} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}$$

For c., we are looking for something with (n^2+1) in the denominator. On $L^2(-\pi,\pi)$,

$$f(t) = \sinh t = \sum_{1}^{\infty}\frac{2\sinh\pi}{\pi}\frac{(-1)^{n+1}n}{n^2+1}\sin nt$$

so for $f(t) = \sinh t$,

$$\|f\|^2 = \sum_{1}^{\infty}\left|\frac{2\sinh\pi}{\pi}\frac{(-1)^{n+1}n}{n^2+1}\right|^2 = \left(\frac{2\sinh\pi}{\pi}\right)^2\sum_{1}^{\infty}\frac{n^2}{(n^2+1)^2}$$

Now

$$\|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}\sinh^2(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}\cosh(2t) - \frac{1}{2}\right)dt = \frac{\sinh(\pi)\cosh(\pi)-\pi}{\pi}$$

Equating:

$$\frac{\sinh(\pi)\cosh(\pi)-\pi}{\pi} = \left(\frac{2\sinh\pi}{\pi}\right)^2\sum_{1}^{\infty}\frac{n^2}{(n^2+1)^2}$$

$$\pi\frac{\sinh\pi\cosh\pi-\pi}{4\sinh^2\pi} = \sum_{1}^{\infty}\frac{n^2}{(n^2+1)^2}$$

3.4.3

Let D be the unit disk $\{x, y\in\mathbb{R} : x^2+y^2\leq 1\}$ and let $f_n(x,y) = (x+iy)^n$. Show that $\{f_n\}_0^\infty$ is an orthogonal set in $L^2(D)$ and compute $\|f_n\|$ for all n.

Solution: Write $x+iy = e^{i\theta}r$ with $r = \sqrt{x^2+y^2}$. We know from this chapter that an inner product on D is

$$\langle f, g\rangle = \int_D f(x,y)\overline{g(x,y)}dxdy = \int_0^1\int_0^{2\pi} f(r,\theta)\overline{g(r,\theta)}r\,d\theta dr$$

$$f_n(x,y) = (x+iy)^n = r^ne^{in\theta}, \qquad \overline{f_m(x,y)} = (x-iy)^m = r^ne^{-im\theta}$$

$$\langle f_n, f_m\rangle = \int_0^1\int_0^{2\pi} r^{n+m}e^{i(n-m)\theta}r\,d\theta dr = \int_0^1 r^{n+m+1}\left(\int_0^{2\pi}e^{i(n-m)\theta}d\theta\right)dr$$

If $n=m$

$$\int_0^{2\pi}e^{i(n-m)\theta}d\theta = \int_0^{2\pi}d\theta = 2\pi$$

and if $n \neq m$

$$\int_0^{2\pi} e^{i(n-m)\theta} d\theta = \frac{1}{i(n-m)} \left(e^{i(n-m)2\pi} - 1 \right) = \frac{i}{n-m} \left(1 - e^{i(n-m)2\pi} \right) = 0$$

using periodicity. Thus we arrive at

$$\begin{aligned} \langle f_n, f_m \rangle &\neq 0, \qquad \text{if } n \neq m \\ \langle f_n, f_m \rangle &= 0, \qquad \text{if } n = m \end{aligned}$$

What is $\|f_n\|^2$?

3.1:1

Cauchy-Schwarz' inequality and norm convergence gives

$$|\langle f_n - f, g \rangle| \leq \|f_n - f\| \|g\| \to 0 \quad \text{as } n \to \infty$$

The left hand side is

$$|\langle f_n - f, g \rangle| = |\langle f_n, g \rangle - \langle f, g \rangle|$$

So $\|f_n - f\| \to 0$ implies that $|\langle f_n, g \rangle - \langle f, g \rangle| \to 0$, and we are done.

3.1:2

Notice that both $|\|f\| - \|g\||$ and $\|f - g\|$ are non-negative.

$$\|f - g\|^2 = \langle f - g, f - g \rangle = \langle f, f \rangle - \langle f, g \rangle - \langle g, f \rangle + \langle g, g \rangle$$

$$\langle f, g \rangle + \langle g, f \rangle = 2\text{Re}\{\langle f, g \rangle\}$$

$$\|f - g\|^2 = \|f\|^2 - 2\text{Re}\{\langle f, g \rangle\} + \|g\|^2$$

$$|\|f\| - \|g\||^2 = \|f\|^2 - 2\|f\|\|g\| + \|g\|^2$$

Cauchy-Schwarz' inequality and complex algebra gives

$$\text{Re}\{\langle f, g \rangle\} \leq |\langle f, g \rangle| \leq \|f\|\|g\|$$

Collecting results:

$$|\|f\| - \|g\||^2 \leq \|f - g\|^2$$

If $\|f_n - f\| \to 0$ then $|\|f_n\| - \|f\|| \to 0$.

3.5:4

Find the eigenvalues and normalized eigenfunctions for the problem

$$\begin{aligned} &f'' + \lambda f = 0 \quad \text{on } [0, l] \\ &f(l) = 0 \\ &f'(0) = 0 \end{aligned}$$

Solution:

If $\lambda = 0$ then $f(x) = c_0 + c_1 x$ but the boundary conditions give $c_0 = c_1 = 0$. Let $\nu^2 = \lambda$ and assume that $\lambda > 0$. The general solution of the differential equation $f'' + \nu^2 f = 0$ is

$$f(x) = a \cos \nu x + b \sin \nu x, \qquad \nu^2 = \lambda$$

Use conditions! We have

$$f'(0) = b\nu \implies b = 0$$

and

$$f(l) = a \cos \nu l = 0 \implies \nu l = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

Take $a = 1$, and let $n \in \mathbb{N}$ since the actual eigenvalues are ν^2. Thus

$$f(x) = \cos\left(\frac{\pi}{l}\left[\frac{1}{2} + n\right] x\right), \quad \lambda = \sqrt{\frac{\pi}{l}\left[\frac{1}{2} + n\right]}, \quad n \in \mathbb{N}$$

Now assume $\lambda < 0$ and let $\lambda = -\mu^2$.

$$f'' - \mu^2 f = 0 \implies f(x) = ae^{\mu x} + be^{-\mu x}$$

Use conditions! We have

$$f'(0) = \mu(a - b) = 0 \implies a = b$$

and

$$f(l) = ae^{\mu l} + be^{-\mu l} = a\left(e^{\mu l} + e^{-\mu l}\right) = 0$$

but $\cosh(\mu l) \neq 0$ and hence $a = 0$. So if $\lambda < 0$ then no eigenfunctions exist (except $f = 0$).

EÖ 23

Bestäm samtliga egenvärden och egenfunktioner till Sturm-Liouville-problemet

$$\begin{cases} f'' + \lambda f = 0, & 0 < x < l \\ f(0) = f'(0), & f(l) + 2f'(l) = 0. \end{cases}$$

Solution: If $\lambda = 0$ then $f(x) = c_0 + c_1 x$ but the boundary conditions give $c_0 = c_1 = 0$. Let $\nu^2 = \lambda$ and assume that $\lambda > 0$. The general solution of the differential equation $f'' + \nu^2 f = 0$ is

$$f(x) = a \cos \nu x + b \sin \nu x, \qquad \nu^2 = \lambda$$

Use conditions! We have $f'(x) = -\nu a \sin \nu x + \nu b \cos \nu x = \nu(-a \sin \nu x + b \cos \nu x)$, so

$$f(0) = f'(0) \implies a = \nu b$$

and

$$f(l) + 2f'(l) = 0 \iff a \cos \nu l + b \sin \nu l + 2\nu(-a \sin \nu l + b \cos \nu l) = 0$$

so

$$\begin{aligned} 0 = 3\nu \cos \nu l + \sin \nu l - 2\nu^2 \sin \nu l = \nu \cos \nu l + (1 - 2\nu^2) \sin \nu l \\ = (1 - 2\nu^2) \cos \nu l \left(\frac{3\nu}{1 - 2\nu^2} + \tan \nu l \right) \end{aligned}$$

Notice that if $\nu^2 = 1/2$ or $\cos \nu l = 0$ the original equation is not fulfilled. Restricting to $\nu > 0$ as before,

$$0 = \frac{3\nu}{1 - 2\nu^2} + \tan \nu l$$

will have solutions $\{\nu_n\}$ for some parameters (like $l = 1$). Conclusion: eigenfunctions $\cos \nu_n x$ and $\sin \nu_n x$, where $\{\nu_n\}$ are solutions to the equation above, exist.

For $\lambda = -\mu^2 < 0$ we get

$$f(x) = \tilde{a} e^{\mu x} + \tilde{b} e^{-\mu x} = a \frac{e^{\mu x} + e^{-\mu x}}{2} + b \frac{e^{\mu x} - e^{-\mu x}}{2} = a \cosh \mu x + b \sinh \mu x$$

and $f'(x) = \mu a \sinh \mu x + \mu b \cosh \mu x$ so

$$f(0) = f'(0) \implies a = \mu b$$

and

$$f(l) + 2f'(l) = 0 \iff a \cosh \mu l + b \sinh \mu l + 2\mu(a \sinh \mu l + b \cosh \mu l) = 0$$

so

$$0 = 3\mu \cosh \mu l + \sinh \mu l + 2\mu^2 \sinh \mu l = (1 + 2\mu^2) \cosh \mu l \left(\frac{3\mu}{1 + 2\mu^2} + \tanh \mu l \right)$$

Observe that $\cosh x > 0$ for all real x. So are there solutions to

$$\tanh(\mu l) + \frac{3\mu}{1 + 2\mu^2} = 0 \ ?$$

No. Both terms are either positive or negative, simultaneously.

4.2:1

This problem concerns heat flow in a rod on the interval $[0, l]$; it is assumed that heat can enter or leave the rod only at the ends.

Suppose the end $x = 0$ is held at temperature zero while the end $x = l$ is insulated.

(a) Find a series expansion for the temperature $u(x, t)$ given the initial temperature $f(x)$.

(b) What is $u(x,t)$ when $f(x) = 50$ for all x?

Solution: Solve

$$u_t = ku_{xx}, \quad u(0,t) = 0, \quad u_x(l,t) = 0, \quad u(x,0) = f(x)$$

Set $u(x,t) = X(x)T(t)$ to obtain

$$\frac{T'}{T} = k\frac{X''}{X}, \quad X(0) = 0, \quad X'(l) = 0$$

So

$$T(t) = T(0)e^{-k\lambda t}$$

and $X'' + \lambda X = 0$ with $X(0) = 0$, thus $\lambda = \nu^2$ and

$$X(x) = a \sin \nu x,$$

and

$$0 = X'(l) = \nu \cos \nu l \implies \nu l = \frac{\pi}{2} + n\pi, \quad n \in \{0, 1, 2, 3, ...\}$$

so the conclusion is $u(x,t) = \sum A_n \sin(\nu_n x)e^{-k\nu^2 t}$, where ν_n is the sequence above. Notice that if $\lambda = -\mu^2 < 0$ then $X(x) = a \sinh(\mu x)$ but $X'(l) = \mu \cosh \mu l > 0$. At $t = 0$

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \sin(\nu_n x)$$

Taking inner products (with proper normalization),

$$\langle f, \sin \nu_m \cdot \rangle = A_m$$

(Here, the dot after ν_m indicates where the x should go. Think about it: We don't write $f(x)$ but only f for the same reason, namely that f is a function and $f(x)$ is a function value at point x.)

The full solution is

$$u(x,t) = \sum_{n=0}^{\infty} \langle f, \sin \nu_n \cdot \rangle e^{-k\nu_n^2 t} \sin \nu_n x$$

If $f = 50$ then

$$\langle f, \sin \nu_n \cdot \rangle = \frac{2}{l} \int_0^l 50 \sin \nu_n t dt = \frac{100}{l} \frac{1}{\nu_n} = \frac{200}{\pi(2n+1)}$$

so then

$$u(x,t) = \frac{200}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{-k\nu_n^2 t} \sin \nu_n x$$

Open questions

Going back to sum computations, why is the following not working? Write

$$t^2 - \frac{\pi^2}{3} = \sum_1^\infty \frac{4(-1)^n}{n^2}\cos(nt)$$

so by Parseval's equality

$$\|t^2 - \frac{\pi^2}{3}\|^2 = \sum_1^\infty \left|\frac{4(-1)^n}{n^2}\right|^2$$

Now

$$\|t^2 - \frac{\pi^2}{3}\|^2 = \int_{-\pi}^{\pi}(t^2 - \frac{\pi}{3})^2 dx = \int_{-\pi}^{\pi}(t^4 - 2t^2\frac{\pi}{3} + \frac{\pi^2}{9})dx = \frac{2}{45}\pi^3(5 - 10\pi + 9\pi^2)$$

and

$$\sum_1^\infty \left|\frac{4(-1)^n}{n^2}\right|^2 = 16\sum_1^\infty \frac{1}{n^4}$$

so Parseval's equality says that

$$\sum_1^\infty \frac{1}{n^4} = \frac{1}{16}\frac{2}{45}\pi^3(5 - 10\pi + 9\pi^2) = \frac{1}{360}\pi^3(5 - 10\pi + 9\pi^2)$$