Chalmers/GU
Mathematics

**EXAM SOLUTION**

# TMA947/MMG621
# NONLINEAR OPTIMISATION

**Date:** 18–11–01
**Examiner:** Michael Patriksson

Note: the solutions presented here are brief in relation to the requirements on your answers, in particular regarding your motivations.

## Question 1

(the simplex method)

**(2p)** a) Rewrite the problem into standard form by letting $x_1 := x_1^+ - x_1^-$ and adding slack variables $s_1$ and $s_2$ to the left-hand side in the first and second constraint, respectively. Moreover, let $z := -z$ to get the problem on minimization form. Thus, we get the following linear program:

$$\begin{array}{lrcrcrcrcrl} \text{minimize} & z = -x_1^+ & + & x_1^- & - & 4x_2, & & & & & \\ \text{subject to} & x_1^+ & - & x_1^- & + & 3x_2 & + & s_1 & & & = 8, \\ & 2x_1^+ & - & 2x_1^- & + & x_2 & & & - & s_2 & = 4, \\ & x_1^+, & & x_1^-, & & x_2, & & s_1, & & s_2 & \geq 0. \end{array}$$

Introducing the artificial variable $a$, phase I gives the problem

$$\begin{array}{lrcrcrcrcrcrl} \text{minimize} & w = a, & & & & & & & & & & & \\ \text{subject to} & x_1^+ & - & x_1^- & + & 3x_2 & + & s_1 & & & & & = 8, \\ & 2x_1^+ & - & 2x_1^- & + & x_2 & & & - & s_2 & + & a & = 4, \\ & x_1^+, & & x_1^-, & & x_2, & & s_1, & & s_2, & & a & \geq 0. \end{array}$$

Using the starting basis $(s_1, a)^T$ gives

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{N} = \begin{pmatrix} 1 & -1 & 3 & 0 \\ 2 & -2 & 1 & -1 \end{pmatrix}, \boldsymbol{x}_B = \begin{pmatrix} 8 \\ 4 \end{pmatrix}, \boldsymbol{c}_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{c}_N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The reduced costs, $\bar{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N}$, for this basis is $\bar{\boldsymbol{c}}_N^T = \begin{pmatrix} -2, & 2, & -1, & 1 \end{pmatrix}$, which means that $x_1^+$ enters the basis. The minimum ratio test implies that $a$ leaves.

Thus, we move on to phase II using the basis $(s_1, x_1^+)^T$, and

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \boldsymbol{N} = \begin{pmatrix} -1 & 3 & 0 \\ -2 & 1 & -1 \end{pmatrix}, \boldsymbol{x}_B = \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \boldsymbol{c}_B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \boldsymbol{c}_N = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}.$$

The new reduced costs are $\bar{\boldsymbol{c}}_N^T = \begin{pmatrix} 0, & -3.5, & -0.5 \end{pmatrix}$ which means that $x_2$ enters the basis. The minimum ratio test implies that $s_1$ leaves.

Updating the basis, now with $(x_1^+, x_2)^T$, gives

$$\boldsymbol{B} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, \boldsymbol{N} = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}, \boldsymbol{x}_B = \begin{pmatrix} 0.8 \\ 2.4 \end{pmatrix}, \boldsymbol{c}_B = \begin{pmatrix} -1 \\ -4 \end{pmatrix}, \boldsymbol{c}_N = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The new reduced costs are $\bar{\mathbf{c}}_N^T = \begin{pmatrix} 0, & 1.4, & 0.2 \end{pmatrix}$. Since the reduced costs are all non-negative, the current BFS is optimal. Returning to the original variables, we obtain $(x_1, x_2) = (0.8, 2.4)^T$ as the optimal solution and $-10.4$ as the optimal value.

**(1p)** b) In the optimal BFS, the reduced cost corresponding to $x_1^-$ is zero. Therefore, we can let $x_1^-$ enter the basis without changing the objective. We do not obtain any leaving variable as minimum ratio implies that the problem is unbounded in that direction. This is simply the increasing $x_1^+$ and increasing $x_1^-$ by the same amount (which can be any positive number). So the problem in standard form does not have a unique optimal solution. But the problem formulated in the original variables does since all these solutions correspond to $(8, 0)^T$, that is due to the reduced cost for $x_2$ and $s_1$ is positive. Replacing one free variable with two positive variables always implies that each solution is non-unique in the sense described above.

---

## (3p) Question 2

(necessary local and sufficient global optimality conditions)

See Propositions 4.22 and 4.23 in the book.

---

## Question 3

(Unconstrained optimization)

**(2p)** a) Set $f(\mathbf{x}) = x_1^2 + 6x_1x_2 + x_2^2$, the Hessian is $\nabla^2 f(\mathbf{x}) = \begin{bmatrix} 2 & 6 \\ 6 & 2 \end{bmatrix}$. Since the multiplier $\gamma = 6$, we get the new matrix $\nabla^2 f(\mathbf{x}) + \gamma \mathbf{I}^2 = \begin{bmatrix} 8 & 6 \\ 6 & 8 \end{bmatrix}$. By $[\nabla^2 f(\mathbf{x}_k) + \gamma_k \mathbf{I}^2]\mathbf{p}_k = -\nabla f(\mathbf{x}_k)$, we can get the search direction $\mathbf{p}_0$ for $\mathbf{x}_0$ is $(-4, -4)^T$.To determine the step length, we perform Armijo line search. Start from $\alpha = 1$, $f(\mathbf{x}_0 + \alpha \mathbf{p}_0) - f(\mathbf{x}_0) \leq \mu \alpha \nabla f(\mathbf{x}_o)^T \mathbf{p}_k$ is not fulfilled. Then take $\alpha = 1/2$, the inequality is fulfilled. So $\mathbf{x}_1 = \mathbf{x}_0 + \alpha \mathbf{p}_0 = (5, 5)^T$.

**(1p)** b) Let $x_1 = -\lambda$ and $x_2 = \lambda$, when $\lambda \to \infty$, the objective function value will go to $-\infty$, which means the problem is unbounded, so there is no global optimal. So Newtons Levenberg-Marquardt method can't converge to a global optimal.

---

## Question 4

(KKT conditions)

**(1p)** a) Affine CQ doesn't hold since by definition it only applies to affine constraints.

**(1p)** b) Slater CQ requires inequality constraints to be level sets of convex functions $g(x) \leq 0$. However, $g = 25 - (x_1 + 3)^2 - (x_2 - 4)^2$ is a strictly concave function, this since the hessian, $\nabla^2 g(x) = -2\mathbf{I}$, is negative definte.

**(1p)** c) Three constraints are active in the point $\bar{\mathbf{x}}^{\mathrm{T}} = (0, 0)$, hence the gradients of these constraints should be linearly independent for LICQ to be satisfied. But this is impossible since the dimension of the space is two and as the number of active constraints is three. Could also be verified by computing the gradients $\nabla g_1(\bar{\mathbf{x}})^{\mathrm{T}} = (-2, 0)$, $\nabla g_2(\bar{\mathbf{x}})^{\mathrm{T}} = (6, -8)$, and $\nabla g_3(\bar{\mathbf{x}})^{\mathrm{T}} = (6, 8)$. Then see that $6g_1(\bar{\mathbf{x}}) + g_2(\bar{\mathbf{x}}) + g_3(\bar{\mathbf{x}}) = \mathbf{0}$, i.e., they are linearly dependent and thus viloating LICQ.

## (3p) Question 5

(modelling) Sets, let

- $\mathcal{K} = \{\texttt{mb}, CPU, GPU, PSU, \texttt{hd}\}$, be the set of components.
- $\mathcal{K}_- = \mathcal{K} \setminus \{\texttt{mb}\}$.
- $\mathcal{M}_k$ be the set of acceptable variants for component $k \in \mathcal{K}$.
- $\mathcal{P}_{mk} \subseteq \mathcal{M}_k$ be the set of variants of component $k \in \mathcal{K}_-$, incompatible with motherboard $m \in \mathcal{M}_{\texttt{mb}}$.

Parameters, let

- $c_{km}$ be the cost of model $m \in \mathcal{M}_k$ of component $k \in \mathcal{K}$.
- $p_{km}$ be the power of model $m \in \mathcal{M}_k$ of component $k \in \mathcal{K}$. (PSU has negative power)

Variables, let

- $x_{km}$ be the binary choice of buying model $m \in \mathcal{M}_k$ of component $k \in \mathcal{K}$.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{k \in \mathcal{K}} \sum_{m \in \mathcal{M}_k} c_{km} x_{km}, && && (1)\\
\text{s.t.} \quad & \sum_{k \in \mathcal{K}} \sum_{m \in \mathcal{M}_k} p_{km} x_{km} \leq 0, && && (2)\\
& x_{\texttt{mb},m} + x_{kl} \leq 1, && l \in \mathcal{P}_{mk},\ m \in \mathcal{M}_{\texttt{mb}},\ k \in \mathcal{K}_-, && (3)\\
& \sum_{m \in \mathcal{M}_k} x_{km} = 1, && k \in \mathcal{K}, && (4)\\
& x_{km} \in \{0,1\}, && m \in \mathcal{M}_k, k \in \mathcal{K}. && (5)
\end{aligned}
$$

(1) mininmize the costs, (2) must have excess power, (3) cannot choose a model incompatible with the MB, (4) each component needs to be installed, and (5) the choices are binary.

## Question 6

(true or false)

**(1p)** a) False, the penalty goes to zero; allowing the optimal solution to be arbitrarily close to the boundary of the feasible set. Letting the penalty go to infinity would imply the original objective function redudant.

**(1p)** b) False, take two points $x_1 = (-2, 0)^T$ and $x_2 = (0, 5)^T$, then $x_1$ and $x_2$ are in $S$. But one convex combination of $x_1$ and $x_2$: $1/2 * (x_1 + x_2) = (-1, 5/2)$ is not in $S$, since $5/2 * sin(-1) < 0$.

**(1p)** c) False, complementary slackness conditions are not fulfilled ($\mu_2^* g_2(x^*) = -2 \neq 0$).

---

## (3p) Question 7

linear programming duality Suppose, for example, that $X$ is bounded. Then, there exists a bounded optimal solution for every value of the coefficient vector $\boldsymbol{c}$. Therefore, its dual must also have bounded optimal solutions for every value of $\boldsymbol{c}$. It then follows that the dual problem must have feasible solutions for every $\boldsymbol{c}$. Consider the cone

$$C := \{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{0}, \quad \boldsymbol{y} \geq \boldsymbol{0}\}.$$

By the Representation Theorem, the set $Y$ is bounded if and only if $C$ contains only the zero vector. Since the dual problem must have feasible solutions for every $\boldsymbol{c}$, choose $\boldsymbol{c} = -\boldsymbol{e}$, where $\boldsymbol{e}$ if the $m$-vector of ones. Then we have that the set

$$\{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq -\boldsymbol{e}, \quad \boldsymbol{y} \geq \boldsymbol{0}\},$$

is non-empty. Clearly, any of its members are non-zero, and moreover they belong to the cone $C$. Hence, $C$ does not only contain the zero vector, ans so $Y$ is unbounded.

The case when one assumes that $Y$ is bounded is treated similarly.

---