Problems IV on Calculus of Variations

Matematik Lth Spring 2020

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**Problem 1** Minimize the functional

\[
J[y] = \int_{0}^{2} y'(x)(x + y'(x)) \, dx
\]

over all \( y \in D^1 \) with \( y(0) = y(2) = 0 \).

**Problem 2** Consider the functional

\[
J[y] = \int_{a}^{b} f(y(x), y'(x)) \, dx
\]

and the boundary condition \( y(a) = y(b) \). Assume that the function \( y' \mapsto f_y(y, y') \) is strictly monotonic for all \( y \). Prove that any extremal can be extended to a periodic \( C^1 \) function.

**Problem 3**

1. Use the corner conditions to find all \( D^1 \) extremals for the functional (cmp. Problem I.6)

\[
J[y] = \int_{0}^{2} y^2(x)(1 - y'(x))^2 \, dx
\]

with the boundary conditions \( y(0) = 0 \) and \( y(2) = 1 \).

2. (Hard) Find all \( D^1 \) extremals for the functional above with the boundary conditions \( y(0) = 0 \) and \( y(2) = A \) for all real \( A \).
Problem 4  For the problem of minimizing

\[ J[y] = \int_a^b (y'(x)^4 - y'(x)^2) \, dx \]

subject to \( y(a) = A \) and \( y(b) = B \),

1. Find a condition that determines that a broken extremal with exactly one corner exists.

2. Assuming the condition holds, find all such extremals.

Problem 5 (Hard)  Consider the simplest problem of calculus of variations for the functional

\[ J[y] = \int_a^b f(x, y(x), y'(x)) \, dx \]

with \( y(a) \) and \( y(b) \) being fixed. Assume that \( f \in C^2 \). Prove that

\[ \inf_{y \in D^1} J[y] = \inf_{y \in C^1} J[y]. \]

Hint: consider \( y \in D^1 \) with just one corner at \( c \) and show that it can be approximated by \( \hat{y} \in C^1 \) such that \( J[y] \) and \( J[\hat{y}] \) can be done arbitrarily close. For example, consider \( y'(x) = g(x) + A\theta(x - c) \) where \( g \) continuous and \( \theta \) is Heaviside function and construct an explicit approximation, then estimate \( J \) via Taylor for \( f \).