FOURIER ANALYSIS & METHODS 2020.02.19

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ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Applications of the Fourier transform

We will use the Fourier transform to solve both the homogeneous heat equation as well as the inhomogeneous heat equation. To do this, we briefly recall an important calculation. We would like to compute

$$\int_{\mathbb{R}} e^{-x^2} dx.$$

There is a beautiful trick for doing this calculation. Here is where the idea originates. If this integral were

$$\int_{\mathbb{R}} xe^{-x^2} dx$$

we would know how to compute it. So we would like to be integrating against xdx not just dx. When do we have something like xdx? We have something of this form when we are working in polar coordinates in $\mathbb{R}^2$, because there we have $rdrd\theta$. So, we could compute the integral

$$\int_{\mathbb{R}^2} e^{-r^2} rdrd\theta = 2\pi \int_0^\infty e^{-r^2} rdr = 2\pi \frac{e^{-r^2}}{-2}\Big|_0^\infty = \pi.$$

On the other hand

$$\int_{\mathbb{R}^2} e^{-r^2} rdrd\theta = \int_{\mathbb{R}^2} e^{-x^2-y^2} dxdy = \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy = \left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2.$$

Thus

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}.$$

1.1. **Homogeneous heat equation.** We wish to solve:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0, & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = v(x), \end{cases}$$

where our initial data v is assumed to be bounded, continuous, and also in $\mathcal{L}^2(\mathbb{R})$.

> Idea: Fourier transform the PDE with respect to the x variable, because $x \in \mathbb{R}$, whereas $t > 0$, but the Fourier transform integrates over all of $\mathbb{R}$, thus x is the wise choice.

We obtain
$$\hat{u}_t(\xi,t) - \hat{u}_{xx}(\xi,t) = 0.$$
Now, we use the theorem which gave us the properties of the Fourier transform. It says that if we take the Fourier transform of a derivative, $\hat{f'}(\xi) = i\xi\hat{f}(\xi)$. Using this twice,
$$\hat{u}_{xx}(\xi,t) = -\xi^2\hat{u}(\xi,t).$$
Now, those of you who are picky about switching limits may not like this, but it is in fact rigorously valid:
$$\partial_t\hat{u}(\xi,t) + \xi^2\hat{u}(\xi,t) = 0.$$
Hence
$$\partial_t\hat{u}(\xi,t) = -\xi^2\hat{u}(\xi,t).$$
This is a first order homogeneous ODE for u in the t variable. We can solve it!!! We do that and get
$$\hat{u}(\xi,t) = e^{-\xi^2 t}c(\xi).$$
The constant can depend on ξ but not on t. To figure out what the constant should be, we use the IC:
$$\hat{u}(\xi,0) = \hat{v}(\xi) \implies c(\xi) = \hat{v}(\xi).$$
Thus, we have found
$$\hat{u}(\xi,t) = e^{-\xi^2 t}\hat{v}(\xi).$$
Now, we use another property of the Fourier transform which says
$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$
So, if we can find a function whose Fourier transform is $e^{-\xi^2 t}$, then we can express u as a convolution of that function and v. So, we are looking to find
$$g(x,t) \text{ such that } \hat{g}(x,t) = e^{-\xi^2 t}.$$
We use the FIT:
$$g(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}e^{-\xi^2 t}d\xi.$$
We can use some complex analysis to compute this integral. To do this, we shall complete the square in the exponent:
$$-\xi^2 t + ix\xi = -\left(\xi\sqrt{t} - \frac{ix}{2\sqrt{t}}\right)^2 - \frac{x^2}{4t}.$$
Therefore we are computing
$$\int_{\mathbb{R}} \exp\left(-\left(\xi\sqrt{t} - \frac{ix}{2\sqrt{t}}\right)^2 - \frac{x^2}{4t}\right)d\xi.$$
Using a contour integral, we can in fact ignore the imaginary part. To see this, first note that we are integrating with respect to ξ, so we can for the moment just consider:
$$\int_{-\infty}^{\infty} \exp\left(-\left(\xi t - \frac{ix}{2\sqrt{t}}\right)^2\right)d\xi.$$
We draw a box. The box has vertices in the complex plane at the points $\pm R$ and $\pm R + \frac{ix}{2\sqrt{t}}$. The integrand above is holomorphic for all ξ inside this box. Therefore the integral around the boundary of the box is zero. When $\xi = \pm R$, the integrand is very small, thus the integrals on the vertical sides of the box tend to zero. Hence

the integrals along the two horizontal sides of the box are also adding up to zero, which shows that

$$\int_{-\infty}^{\infty} \exp\left(-\left(\xi t - \frac{ix}{2\sqrt{t}}\right)^2\right) d\xi = \int_{-\infty}^{\infty} \exp(-\xi^2 t^2) d\xi.$$

So, we compute (using a change of variables to $y = \xi\sqrt{t}$ so $t^{-1/2}dy = d\xi$)

$$\int_{\mathbb{R}} e^{-\xi^2 t} d\xi = \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{-y^2} dy = \frac{\sqrt{\pi}}{\sqrt{t}}.$$

Hence,

$$\int_{\mathbb{R}} \exp\left(-\left(\xi\sqrt{t} - \frac{ix}{2\sqrt{t}}\right)^2 - \frac{x^2}{4t}\right) d\xi = \frac{\sqrt{\pi}}{\sqrt{t}} e^{-\frac{x^2}{4t}}.$$

Recalling the factor of $1/(2\pi)$ we have

$$g(x,t) = \frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{t}} e^{-\frac{x^2}{4t}} = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.$$

Hence the solution is

$$u(x,t) = g * v(x) = \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi t}} e^{-(x-y)^2/(4t)} v(y) dy.$$

Exercise 1. *Verify that for all $x \in \mathbb{R}$ and $t > 0$ our solution satisfies the homogeneous heat equation.*

Question 1. *Why is our solution equal to v when $t = 0$?*

If we naively set $t = 0$, we obtain an expression that does not make sense. So, how do we know that this expression indeed gives us our initial data at $t = 0$? We use the big bad convolution approximation theorem! Consider the function

$$\varphi(x) = \frac{e^{-x^2/4}}{2\sqrt{\pi}}.$$

This function satisfies

$$\int_{\mathbb{R}} \varphi(x) dx = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2} dz = 1,$$

using the change of variables $z = \frac{x}{2}$. This function satisfies the hypotheses of the theorem (the so-called g function). We have assumed that v is bounded. Therefore the convolution approximation theorem says that

$$\lim_{\varepsilon \downarrow 0} \varphi_\varepsilon \star v(x) = v(x) \quad \forall x \in \mathbb{R}.$$

Let's re-name ε to $\sqrt{t}$, so that

$$\lim_{\sqrt{t} \downarrow 0} \varphi_{\sqrt{t}} \star v(x) = v(x).$$

Let's write out the

$$\varphi_{\sqrt{t}} \star v(x) = \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{(2\sqrt{t})^2}}}{2\sqrt{\pi}\sqrt{t}} v(y) dy.$$

The theorem says

$$\lim_{\sqrt{t}\downarrow 0} \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{(2\sqrt{t})^2}}}{2\sqrt{\pi}\sqrt{t}} v(y) dy = \lim_{t \downarrow 0} \int_{\mathbb{R}} \frac{e^{-(x-y)^2/(4t)}}{2\sqrt{\pi t}} v(y) dy = \lim_{t\downarrow 0} u(x,t) = v(x) \quad \forall x \in \mathbb{R}.$$

We therefore understand

$$u(x,0) := \lim_{t \downarrow 0} u(x,t) = v(x) \forall x \in \mathbb{R}.$$

With some abstract uniqueness theory, beyond the scope of this humble course, we could also prove that our solution $u(x,t)$ is the unique solution to the heat equation which has initial data equal to $v(x)$ and which is in $\mathcal{L}^2$ for all $t > 0$.

1.2. **Inhomogeneous heat equation.** If you have an inhomogeneous IVP for the heat equation, here are two ways to deal with that:

(1) If the inhomogeneity is *time independent,* look for a steady state solution to solve the inhomogeneous equation. Then, solve the homogeneous equation, but change your initial data. If f is your steady state solution and v was your initial data (before f came along), solve the IVP for the homogeneous heat equation with IC $v - f$ rather than just v.
(2) If the inhomogeneity is *time dependent,* you can try to solve by Fourier transforming the whole PDE.

Since we know how to do the first type of example, let us consider the second type of example. We want to solve an inhomogeneous heat equation on $\mathbb{R}$:

$$u_t - u_{xx} = G(x,t), \quad u(x,0) = v(x) \text{ is continuous, bounded, and in } \mathcal{L}^2.$$

Let's try the Fourier transform method:

$$\partial_t \hat{u}(\xi,t) + \xi^2 \hat{u}(\xi,t) = \hat{G}(\xi,t).$$

This is a first order ODE. If you are a CHEMIST, then you did the special extra part of the course and actually learned how to solve this ODE in t. Pretty cool. To see how this works, treat ξ like a constant, and write

$$f'(t) + \xi^2 f(t) = \hat{G}(\xi,t).$$

The mμthod says to first compute

$$e^{\int \xi^2 dt} = e^{\xi^2 t}.$$

Next compute

$$\int e^{\xi^2 t} \hat{G}(\xi,t) dt.$$

Then, the solution is

$$\frac{\int e^{\xi^2 t} \hat{G}(\xi,t) dt + C(\xi)}{e^{\xi^2 t}} = e^{-\xi^2 t} \int e^{\xi^2 s} \hat{G}(\xi,s) ds + C(\xi) e^{-\xi^2 t}.$$

We would like the initial condition to be satisfied, so when $t = 0$ we should obtain that this is equal to the Fourier transform of the initial data,

$$\hat{v}(\xi).$$

We are free to choose any primitive function of $e^{-\xi^2 s} \hat{G}(\xi,s)$. It is very convenient to choose the one which vanishes when $t = 0$, namely

$$\int_0^t e^{-\xi^2 s} \hat{G}(\xi,s) ds.$$

Then to obtain the initial condition, we just let $C(\xi) = \hat{v}(\xi)$. Thus, our Fourier transformed solution is

$$e^{-\xi^2 t} \int_0^t e^{-\xi^2 s} \hat{G}(\xi, s) ds + \hat{v}(\xi) e^{-\xi^2 t}.$$

We need to figure out from whence this Fourier transform came (equivalently, invert the Fourier transform). This is a linear process, so we can deal with each piece separately and then add them. Well, the second part we did last time. We saw that the Fourier transform of

$$\frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} v(y) dy$$

is

$$\hat{v}(\xi) e^{-\xi^2 t}.$$

Similarly, let's look at the first part. It is

$$e^{-\xi^2 t} \int_0^t e^{\xi^2 s} \hat{G}(\xi, s) ds = \int_0^t e^{-\xi^2 (t-s)} \hat{G}(\xi, s) ds.$$

By the same calculations, the Fourier transform of

$$\frac{1}{2\sqrt{\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} G(y, s) dy = e^{-\xi^2 (t-s)} \hat{G}(\xi, s).$$

Yet again playing switch-a-roo with limits[1],

$$\mathcal{F}\left(\int_0^t \frac{1}{2\sqrt{\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} G(y, s) dy ds \right)(\xi) = \int_0^t e^{-\xi^2 (t-s)} \hat{G}(\xi, s) ds.$$

Therefore, our full solution is

$$\int_0^t \frac{1}{2\sqrt{\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} G(y, s) dy ds + \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}} v(y) dy.$$

This solution satisfies our initial data because

$$\lim_{t \downarrow 0} \int_0^t \frac{1}{2\sqrt{\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} G(y, s) dy ds = 0,$$

and just as in the homogeneous heat equation, we have by the convolution approximation theorem that

$$\lim_{t \downarrow 0} \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}} v(y) dy = v(x) \quad \forall x \in \mathbb{R}.$$

1.3. **Computing tricky integrals (sometimes you can compute integrals that computers cannot!)** The following is a very useful observation:

$$\hat{f}(0) = \int_{\mathbb{R}} f(x) dx.$$

So, if you have the integral of a function, this is equal to the value of its Fourier transform at $\xi = 0$. So, if you can look up the Fourier transform of the function, like in Beta or Folland, then to compute the integral, no need for fancy contour integrals, simply pop $\xi = 0$ into the Fourier transform.

[1] Trust me!

Here is an example:

$$\text{compute: } \int_{\mathbb{R}} \frac{1}{x^2+9} dx.$$

We see this is # 10 in Folland's TABLE 2. On the right side, we get the Fourier transform (with $a = 3$) is given by

$$\frac{\pi}{3} e^{-3|\xi|}.$$

So, this integral is the Fourier transform with $\xi = 0$, hence the value of the integral is

$$\frac{\pi}{3}.$$

That was pretty easy right? For something more complicated, you could have say

$$\int_{\mathbb{R}} f(x)g(x)dx,$$

with some icky functions f and g (see extra övning # 9). Now, you can use that the Fourier transform of a product is

$$(2\pi)^{-1}(\hat{f} * \hat{g})(\xi).$$

Hence, what you have above is

$$\int_{\mathbb{R}} f(x)g(x)dx = \int_{\mathbb{R}} e^{-i(0)x} f(x)g(x)dx = (2\pi)^{-1}(\hat{f} * \hat{g})(0).$$

So, if the Fourier transforms of these functions are somewhat better than the functions f and g, then the stuff on the right could be nicely computable and give you the integral on the left. Try # 9 to see how this works. (If you get stuck, Team Fourier is here to help! Just ask us!)

As another example, there is extra exercise number 10. It says you know the Fourier transform of $f(t)$ is $\frac{1}{|w|^3+1}$. We are then asked to compute

$$\int_{\mathbb{R}} |f * f'|^2 dt.$$

By the Plancharel theorem,

$$\int_{\mathbb{R}} |f * f'|^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f * f'}|^2 dt.$$

Now we use the theorem on the properties of the Fourier transform which says

$$\widehat{f * f'}(\xi) = \hat{f}(\xi)\widehat{f'}(\xi).$$

Now we use that same theorem to say that

$$\widehat{f'}(\xi) = i\xi \hat{f}(\xi).$$

So, the stuff on the right is

$$\frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi) i\xi \hat{f}(\xi)|^2 d\xi.$$

We are given what the Fourier transform is, so we put it in there:

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\xi^2}{(|\xi|^3+1)^4} d\xi.$$

Now this isn't so terrible. It's an even function so this is

$$\frac{1}{\pi}\int_0^\infty \frac{\xi^2}{(\xi^3+1)^4}d\xi.$$

It just so happens that the derivative of

$$\frac{1}{(\xi^3+1)^3} \text{ is } \frac{-9\xi^2}{(\xi^3+1)^4},$$

so

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{\xi^2}{(\xi^3+1)^4}d\xi = \frac{-1}{9\pi}\left.\frac{1}{(\xi^3+1)^3}\right|_0^\infty = \frac{1}{9\pi}.$$

1.4. **Exercises for the week to be done oneself: hints.**

(1) (Eö 9) Compute (with help of Fourier transform)

$$\int_{\mathbb{R}} \frac{\sin(x)}{x(x^2+1)}dx.$$

Hint: There are disguised zeros and ones hiding all over the place in mathematics. The above is equal to

$$\int_{\mathbb{R}} \frac{\sin(x)}{x(x^2+1)}e^{-i(0)x}dx = \mathcal{F}\left(\frac{\sin x}{x}\frac{1}{x^2+1}\right)(0).$$

So, we now look at Table 2 in Folland, especially item number 8. It says that the Fourier transform of a product is a convolution of the Fourier transforms. So, we apply this to say

$$\mathcal{F}\left(\frac{\sin x}{x}\frac{1}{x^2+1}\right)(0) = \frac{1}{2\pi}\mathcal{F}\left(\frac{\sin x}{x}\right) * \mathcal{F}\left(\frac{1}{x^2+1}\right)(0).$$

Now we use items 10 and 13 from the same table, together with the definition of the convolution, to substitute for the Fourier transforms on the right side:

$$\frac{1}{2\pi}\int_{\mathbb{R}} \pi\chi_1(0-y)\pi e^{-|y|}dy.$$

Recalling what χ_1 means:

$$= \frac{\pi}{2}\int_{-1}^1 e^{-|y|}dy.$$

I leave it to you do compute the integral!

(2) (Eö 67) Compute the Fourier transform of the characteristic function for the interval (a, b) both directly and by using the known case for the interval $(-a, a)$.

Hint: Well, doing it directly we are computing

$$\int_a^b e^{-ix\xi}dx = \begin{cases} b-a & \xi = 0 \\ \frac{i}{\xi}\left(e^{-bi\xi} - e^{-ai\xi}\right) & \xi \neq 0 \end{cases}$$

To do it the other way, it's convenient to introduce some notations:

$$m := \frac{a+b}{2}, \ell := \frac{b-a}{2}.$$

Then our interval is $[m-\ell, m+\ell]$. So we are computing

$$\int_{m-\ell}^{m+\ell} e^{-ix\xi} dx.$$

To make this more familiar let's do a change of variables so that the integral goes from $-\ell$ to ℓ, so we let $t = x - m$, then $dt = dx$, so we are computing

$$\int_{-\ell}^{\ell} e^{-i(t+m)\xi} dt = e^{-im\xi} \int_{-\ell}^{\ell} e^{-it\xi} dt = e^{-im\xi} \hat{\chi}_{[-\ell,\ell]}(\xi).$$

So now for the Fourier transform of the characteristic function of the interval, that is the function $\chi_{[-\ell,\ell]}$ we can use the item 12 in Table 2 of Folland. With a little algebraic manipulations, one can show that these both roads lead to the same answer.

(3) (7.2.8) Given $a > 0$ let $f(x) = e^{-x}x^{a-1}$ for $x > 0$, $f(x) = 0$ for $x \le 0$. Show that $\hat{f}(\xi) = \Gamma(a)(1+i\xi)^{-a}$ where Γ is the Gamma function.

Hint: one is computing

$$\int_0^\infty e^{-x} e^{-ix\xi} x^{a-1} dx = \int_0^\infty e^{-x(1+i\xi)} x^{a-1} dx.$$

On the other hand,

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt.$$

Try doing a substitution to relate these integrals...

(4) (7.2.12) For $a > 0$ let

$$f_a(x) = \frac{a}{\pi(x^2+a^2)}, \quad g_a(x) = \frac{\sin(ax)}{\pi x}.$$

Use the Fourier transform to show that: $f_a * f_b = f_{a+b}$ and $g_a * g_b = g_{\min(a,b)}$.

Hint: The idea is basically repeated use of the items in Folland's Table 2, and using the FIT. First, compute the Fourier transform of $f_a * f_b$ which is $\hat{f}_a(\xi)\hat{f}_b(\xi)$, so you can write this stuff down. You will get something like $e^{-|x|\cdots}$. Next, use the FIT to return to $f_a * f_b$. Note that one way to write the FIT is

$$f(x) = \frac{1}{2\pi} \hat{\hat{f}}(-x).$$

Do something similar for the second one...

(5) (Eö 6.d,e) Compute the Fourier transform of:

$$e^{-a|t|} \sin(bt), \quad (a, b > 0), \quad \frac{t}{t^2+2t+5}.$$

Hint: I might deal with the first one by splitting up the sine into its complex exponentials, using definition of Fourier transform, and just directly integrating. As for the second one, note that $t^2 + 2t + 5 = (t+1)^2 + 4$. Do a substitution in the definition of the Fourier transform, let $x = t+1$. Then use item 10 on Folland's Table 2.

(6) (Eö 15) Find a solution to the equation

$$u(t) + \int_{-\infty}^{t} e^{\tau - t} u(\tau) d\tau = e^{-2|t|}.$$

Hint: This is a tricky one! First turn the integral into a convolution. How to do that? Try using $\Theta(\tau)e^{-|\tau|}$. Write out the convolution of that function together with $u(\tau)$. Next, Fourier transform both sides of the equation. So you will get

$$\hat{u}(\xi) + (\widehat{\Theta(\tau)e^{-|\tau|}})(\xi)\hat{u}(\xi) = \widehat{e^{-2|t|}}(\xi).$$

Compute the Fourier transforms of everything except u. Solve the equation for $\hat{u}(\xi)$. Then use the FIT. When you use the FIT, if you do it using contour integrals and the residue, you will need to think about the cases $x > 0$ and $x < 0$ separately. For $x > 0$ the up-rainbow will work. For $x < 0$ the down-rainbow will work.

(7) (Eö 11) For the function

$$f(t) = \int_0^2 \frac{\sqrt{w}}{1+w} e^{iwt} dw,$$

compute

$$\int_{\mathbb{R}} f(t)\cos(t)dt, \quad \int_{\mathbb{R}} |f(t)|^2 dt.$$

Hint: This is tricky also. Let me define a new function for us:

$$\phi(w) := \chi_{[0,2]}(w)\frac{\sqrt{2}}{1+w}.$$

Then

$$f(t) = \widehat{\phi}(-t).$$

Oh no she didn't. Yeah. So, for the first one, note that this integral is, expanding the cosine as a sum of complex exponentials

$$\int_{\mathbb{R}} f(t)\cos(t)dt = \frac{1}{2}\left(\hat{f}(1) + \hat{f}(-1)\right).$$

Play around with the FIT and the fact that $f(t) = \widehat{\phi}(-t)$ to figure out the right side. Next, note that

$$\int_{\mathbb{R}} |f(t)|^2 dt = \int_{\mathbb{R}} |\widehat{\phi}(-t)|^2 dt = 2\pi \int_{\mathbb{R}} |\phi(t)|^2 dt.$$

The integral of $|\phi|^2$ is hopefully not that terrible...

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).