FOURIER ANALYSIS & METHODS 2020.02.10

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. SLPs

Recall the definition of a regular SLP:

(1) a formally self-adjoint differential operator

$$L(f) = (rf')' + pf,$$

where r and p are real valued, r, r', and p are continuous, and $r > 0$ on $[a, b]$.

(2) self-adjoint boundary conditions:

$$B_i(f) = \alpha_i f(a) + \alpha_i' f'(a) + \beta_i f(b) + \beta_i' f'(b) = 0, \quad i = 1, 2.$$

The self adjoint condition further requires that the coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i'$ are such that for all f and g which satisfy these conditions

$$r(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

(3) a positive, continuous function w on $[a, b]$.

The SLP is to find all solutions to the BVP

$$L(f) + \lambda w f = 0, \quad B_i(f) = 0, \quad i = 1, 2.$$

The eigenvalues are all numbers λ for which there exists a corresponding non-zero eigenfunction f so that together they satisfy the above equation, and f satisfies the boundary condition.

1.1. SLP example.

Consider the problem

$$(x^2 f')' + \lambda f = 0, \quad f(1) = f(b) = 0, \quad b > 1.$$

Here we have $r(x) = x^2$ and $w(x) = 1$. The equation is:

$$2xf' + x^2 f'' + \lambda f = 0.$$

We shall consider the three cases for λ.

Case $\lambda = 0$: In this case the equation simplifies to

$$x^2 f'' + 2xf' = 0 \implies \frac{f''}{f'} = -\frac{2}{x} \implies (\log(f'))' = -\frac{2}{x} \implies \log(f') = -2\log x \implies f' = e^{-2\log x} = x^{-2}.$$

So, this gives us a solution of the form

$$f(x) = -A\frac{1}{x} + B.$$

Let us verify the boundary conditions. We require $f(1) = 0$ so this means

$$-A + B = 0 \implies B = A.$$

We also require $f(b) = 0$ so this means

$$-A\frac{1}{b} + B = 0 = \frac{-A}{b} + A \implies \frac{A}{b} = A \implies b = 1 \text{ or } A = 0.$$

So since $b > 1$ the only solution here is the zero function which is not an eigenfunction.

Case $\lambda > 0$: We consider the fact that this is an Euler equation, so we look for solutions of the form $f(x) = x^\nu$. Then the equation looks like:

$$x^2(\nu)(\nu-1)x^{\nu-2} + 2x(\nu)x^{\nu-1} + \lambda x^\nu = 0 \iff x^\nu\left(\nu^2 - \nu + 2\nu + \lambda\right) = 0$$

so we need ν to satisfy:

$$\nu^2 + \nu + \lambda = 0.$$

This is a quadratic equation, so we solve it:

$$\nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

So, actually the cases $\lambda > 0$ and $\lambda < 0$ really should split up into whether $\lambda = \frac{1}{4}$ or is larger or smaller. If $\lambda = \frac{1}{4}$, then we are only getting one solution this way, $x^{-1/2}$. To get a second solution we multiply by $\log x$.

Exercise 1. *Plug the function $x^{-1/2}\log x$ into the SLP for the value $\lambda = \frac{1}{4}$. Verify that it satisfy the equation.*

Now, let's see if such a function will satisfy the boundary conditions. We need

$$Ax^{-1/2} + Bx^{-1/2}\log(x)\Big|_{x=1} = 0 \implies A = 0.$$

We also need

$$Bb^{-1/2}\log(b) = 0, \quad b > 1 \implies B = 0.$$

So we only get the zero solution in this case.

When $\lambda < \frac{1}{4}$, solutions are of the form

$$Ax^{\nu_+} + Bx^{\nu_-}, \quad \nu_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

Exercise 2. *Check the boundary conditions. Verify that they are satisfied if and only if $A = B = 0$.*

Finally we consider $\lambda > \frac{1}{4}$. Then we have

$$\nu_\pm = -\frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}} \implies f(x) = \frac{A}{\sqrt{x}}x^{i\sqrt{\lambda-1/4}} + \frac{B}{\sqrt{x}}x^{-i\sqrt{\lambda-1/4}}.$$

Using Euler's formula, this is equivalently expressed as

$$\frac{\alpha}{\sqrt{x}}\cos(\sqrt{\lambda - 1/4}\log x) + \frac{\beta}{\sqrt{x}}\sin(\sqrt{\lambda - 1/4}\log x).$$

Due to the boundary condition at $x = 1$ we must have $\alpha = 0$. So to obtain the other boundary condition, we need

$$\sin(\sqrt{\lambda - 1/4} \log b) = 0 \implies \sqrt{\lambda - 1/4} \log b = n\pi, \quad n \in \mathbb{N}.$$

Hence

$$\lambda = \lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\log b)^2}, \quad f_n(x) = x^{-1/2} \sin\left(\frac{n\pi \log x}{\log b}\right).$$

Note that in general we are not bothering to normalize our eigenfunctions because it is rather tedious and not fundamental to our learning experience in this subject.

2. The theory item on SLPs

There is one theory item about SLPs which one *does* need to be able to prove.

Theorem 1 (Cute facts about SLPs)**.** *Let f and g be eigenfunctions for a regular SLP in an interval $[a, b]$ with weight function $w(x) > 0$. Let λ be the eigenvalue for f and μ the eigenvalue for g. Then:*

(1) $\lambda \in \mathbb{R}$ och $\mu \in \mathbb{R}$;
(2) If $\lambda \neq \mu$, then:

$$\int_a^b f(x)\overline{g(x)}w(x)dx = 0.$$

Proof: By definition we have $Lf + \lambda w f = 0$. Moreover, L is self-adjoint, which similar to matrices guarantees that

$$\langle Lf, f\rangle = \langle f, Lf\rangle.$$

By being an eigenfunction,

$$Lf = -\lambda w f.$$

So combining these facts:

$$\langle Lf, f\rangle = \langle -\lambda w f, f\rangle = -\lambda\langle wf, f\rangle$$
$$= \langle f, Lf\rangle = \langle f, -\lambda w f\rangle = -\overline{\lambda}\langle f, wf\rangle.$$

Since w is real valued,

$$\langle wf, f\rangle = \int_a^b w(x)f(x)\overline{f(x)}dx = \int_a^b |f(x)|^2 w(x)dx,$$
$$\langle f, wf\rangle = \int_a^b f(x)\overline{w(x)f(x)}dx = \int_a^b |f(x)|^2 w(x)dx.$$

Since $w > 0$ and f is an eigenfunction,

$$\int_a^b |f(x)|^2 w(x)dx > 0.$$

So, the equation

$$-\lambda\langle wf, f\rangle = -\lambda \int_a^b |f(x)|^2 w(x)dx = -\overline{\lambda}\langle f, wf\rangle = -\overline{\lambda}\int_a^b |f(x)|^2 w(x)dx$$

implies

$$\lambda = \overline{\lambda}.$$

For the second part, we use basically the same argument based on self-adjointness:

$$\langle Lf, g\rangle = \langle f, Lg\rangle.$$

By assumption

$$\langle Lf, g\rangle = -\lambda\langle wf, g\rangle = -\lambda \int_a^b w(x)f(x)\overline{g(x)}dx.$$

Similarly,

$$\langle f, Lg\rangle = \langle f, -\mu wg\rangle = -\overline{\mu}\langle f, wg\rangle = -\mu\langle f, wg\rangle = -\mu \int_a^b f(x)\overline{g(x)}w(x)dx,$$

since $\mu \in \mathbb{R}$ and $w(x)$ is real. So we have

$$-\lambda \int_a^b w(x)f(x)\overline{g(x)}dx = -\mu \int_a^b f(x)\overline{g(x)}w(x)dx.$$

If the integral is non-zero, then it forces $\lambda = \mu$ which is false. Thus the integral must be zero.

3. Solving PDEs with inhomogeneities: turning a ♡ problem into a ♡♡ problem

Let's consider the problem

$$u(x,0) = \begin{cases} x+\pi, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$$

$$u(-\pi, t) = u(\pi, t) = 0$$

$$u_t(x, 0) = 0, \quad x \in [-\pi, \pi]$$

$$u_{tt}(x,t) - u_{xx}(x,t) = 5 \quad x \in [-\pi, \pi], \quad t > 0.$$

We nickname this problem ♡. For the first time, we have an *inhomogeneous PDE.*

Idea: Deal with a time independent inhomogeneity in the PDE by finding a steady state solution.

The idea is that we look for a function $f(x)$ which depends only on x which satisfies the boundary conditions and also satisfies the inhomogeneous PDE. Since f only depends on x, the PDE for f is

$$-f''(x) = 5 \iff f''(x) = -5.$$

This means that

$$f'(x) = -5x + b \implies f(x) = -\frac{5x^2}{2} + bx + c.$$

Now, we want f to satisfy the boundary conditions. So, we want

$$-\frac{5\pi^2}{2} - b\pi + c = 0 = -\frac{5\pi^2}{2} + b\pi + c.$$

If we subtract these equations, then we see that we need to have $b = 0$. If we add these equations then we see that we need

$$-5\pi^2 + 2c = 0 \implies c = \frac{5\pi^2}{2}.$$

Thus, we have found a solution to

$$-f''(x) = 5, \quad f(\pm\pi) = 0,$$

which is

$$f(x) = -\frac{5x^2}{2} + \frac{5\pi^2}{2}.$$

If we then look for a solution to

$$u(x,0) = \begin{cases} x+\pi, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases} =: v(x)$$

$$u(-\pi,t) = u(\pi,t) = 0$$

$$u_t(x,0) = 0, \quad x \in [-\pi,\pi]$$

$$u_{tt}(x,t) - u_{xx}(x,t) = 0 \quad x \in [-\pi,\pi], \quad t > 0,$$

and we add it to f, we will get

$$u(x,0) + f(x) = v(x) + f(x) \neq v(x).$$

The initial condition gets messed up because of f. So, we need to compensate for this. For that reason, we look for a solution to a new problem:

$$u(x,0) = -f(x) + v(x)$$

$$u(-\pi,t) = u(\pi,t) = 0$$

$$u_t(x,0) = 0, \quad x \in [-\pi,\pi]$$

$$u_{tt}(x,t) - u_{xx}(x,t) = 0 \quad x \in [-\pi,\pi], \quad t > 0.$$

We nickname this new problem ♡♡ because we like it better than ♡. Then, our full solution will be

$$U(x,t) = u(x,t) + f(x).$$

This solution U will then solve ♡. Here it is important to note that when we add u and f, the boundary condition still holds. So, please think about this, because in certain variations on the theme, it could possibly not be true.

Now we can use the techniques we have learned thus far. Separate variables, writing $u(x,t) = X(x)T(t)$. We get the equation

$$T'(t)X(x) - X''(x)T(t) = 0 \iff \frac{T'}{T} = \frac{X''}{X} = \lambda.$$

Since we have super nice BCs for X, we start with the X. We want to solve

$$X''(x) = \lambda X(x), \quad X(-\pi) = X(\pi) = 0.$$

First case: $\lambda = 0$. Then

$$X(x) = ax + b.$$

The BCs say

$$X(-\pi) = -a\pi + b = 0 \implies a\pi = b.$$

Next we need

$$X(\pi) = a\pi + b = 0 \implies b = -a\pi.$$

Combining these,

$$a\pi = -a\pi \implies a = 0 \implies b = 0.$$

So, no solution here because the zero solution doesn't count! Moving right along, let us try

$$\lambda > 0.$$

Then, our solution looks like real exponentials or equivalently sinh and cosh.

HINT: If you interval looks like $[0,l]$, it's probably easiest to work with sinh and cosh because $\sinh(0) = 0$ and $\cosh' = \sinh$. So this will often make things simpler. On the other hand, if you have an interval like $[a, b]$ with a and be not zero, it may be easier to work with the exponentials. So, that's why I'm choosing to do that here. Hence

$$X(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}.$$

The BCs require

$$X(-\pi) = ae^{-\sqrt{\lambda}\pi} + be^{\sqrt{\lambda}\pi} = 0.$$

Let's multiply by $e^{\sqrt{\lambda}\pi}$, to get

$$a + be^{2\sqrt{\lambda}\pi} = 0 \implies a = -be^{2\sqrt{\lambda}\pi}.$$

We check the other BCs

$$X(\pi) = ae^{\sqrt{\lambda}\pi} + be^{-\sqrt{\lambda}\pi} = 0$$

substituting the value of a,

$$-be^{2\sqrt{\lambda}\pi}e^{\sqrt{\lambda}\pi} + be^{-\sqrt{\lambda}\pi} = 0.$$

If $b = 0$ the whole solution is 0, so we assume this is not the case and divide by b. Multiplying by $e^{\sqrt{\lambda}\pi}$ we get

$$-e^{4\sqrt{\lambda}\pi} + 1 = 0 \iff e^{4\sqrt{\lambda}\pi} = 1 \iff 4\sqrt{\lambda}\pi = 0 \iff \lambda = 0,$$

which is a contradiction. So, no solutions lurking over here.

Thus, we consider $\lambda < 0$. Then our solution looks like

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

We need

$$X(-\pi) = a\cos(-\sqrt{|\lambda|}\pi) + b\sin(-\pi\sqrt{|\lambda|}) = 0 = a\cos(\sqrt{|\lambda|}\pi) - b\sin(\sqrt{|\lambda|}\pi),$$

where we use the evenness of cosine and oddness of sine. We also need

$$X(\pi) = a\cos(\sqrt{|\lambda|}\pi) + b\sin(\sqrt{|\lambda|}\pi) = 0.$$

Adding these equations we see that we need

$$a\cos(\sqrt{|\lambda|}\pi) = 0 \implies a = 0 \text{ or } \sqrt{|\lambda|} = \frac{(2k+1)}{2}, \quad k \in \mathbb{Z}.$$

Subtracting these equations we see that we need

$$b\cos(\sqrt{|\lambda|}\pi) = 0 \implies b = 0 \text{ or } \sqrt{|\lambda|} = \frac{2k}{2}, \quad k \in \mathbb{Z}.$$

I know it looks weird but I wrote it this way to make it looks similar to the one with the cosine. Now, the number $\sqrt{|\lambda|}$ can only have one value. It cannot be two different things at the same time. So, we have two types of solutions

$$X_n(x) = \begin{cases} \cos\left(\frac{nx}{2}\right) & n \text{ is odd} \\ \sin\left(\frac{nx}{2}\right) & n \text{ is even.} \end{cases}$$

Here we have

$$\sqrt{|\lambda_n|} = \frac{n}{2}, \quad \lambda_n = -\frac{n^2}{4}.$$

The partner functions,

$$T_n(t) = \alpha_n \cos(\sqrt{|\lambda_n|}x) + \beta_n \sin(\sqrt{|\lambda_n|}x).$$

We shall determine the coefficients using the IC. First, we write

$$u(x,t) = \sum_{n \geq 1} T_n(t) X_n(x).$$

Next, we use the easier of the two ICs, which is

$$u_t(x,0) = 0.$$

So, we also compute

$$u_t(x,t) = \sum_{n \geq 1} T_n'(t) X_n(x).$$

When we plug in 0, we need to have

$$u_t(x,0) = \sum_{n \geq 1} T_n'(0) X_n(x) = 0.$$

So, to get this, we need

$$T_n'(0) = 0 \forall n.$$

By definition of the T_n,

$$T_n'(0) = \beta_n \sqrt{|\lambda_n|}.$$

So, to make this zero, since $\sqrt{|\lambda_n|} \neq 0$, we need

$$\beta_n = 0 \forall n.$$

Hence, our solution looks like

$$u(x,t) = \sum_{n \geq 1} \alpha_n \cos(\sqrt{|\lambda_n|} t) X_n(x).$$

The other IC says

$$u(x,0) = -f(x) + v(x).$$

Since $\cos(0) = 1$, we see that we need

$$-f(x) + v(x) = \sum_{n \geq 1} \alpha_n X_n(x).$$

This means that we need

$$\alpha_n = \frac{\langle -f + v, X_n \rangle}{||X_n||^2} = \frac{\int_{-\pi}^{\pi} (-f(x) + v(x)) X_n(x) dx}{\int_{-\pi}^{\pi} |X_n(x)|^2 dx}.$$

It suffices to just leave α_n like this. As we observed before, our full solution is now

$$U(x,t) = u(x,t) + f(x) = -\frac{5x^2}{2} + \frac{5\pi^2}{2} + \sum_{n \geq 1} \alpha_n \cos(\sqrt{|\lambda_n|} t) X_n(x),$$

with X_n defined as above.

3.1. **Exercises from [1] for the week.**

3.1.1. *To be demonstrated.*

(1) (4.2:5) Solve:
$$u_t = ku_{xx} + e^{-2t}\sin(x),$$
with
$$u(x,0) = u(0,t) = u(\pi,t) = 0.$$

(2) (EO 23) Determine the eigenvalues and eigenfunctions of the SLP:
$$f'' + \lambda f = 0, \quad 0 < x < a,$$
$$f(0) - f'(0) = 0, \quad f(a) + 2f'(a) = 0.$$

(3) (EO 24) Determine the eigenvalues and eigenfunctions of the SLP:
$$-e^{-4x}\frac{d}{dx}\left(e^{4x}\frac{du}{dx}\right) = \lambda u, \quad 0 < x < 1,$$
$$u(0) = 0, \quad u'(1) = 0.$$

(4) (EO 1) A function is 2 periodic with $f(x) = (x+1)^2$ for $|x| < 1$. Expand $f(x)$ in a Fourier series. Search for a 2 periodic solution to the equation
$$2y'' - y' - y = f(x).$$

(5) (4.2.6) Solve:
$$u_t = ku_{xx} + Re^{-ct}, \quad R, c > 0,$$
$$u(x,0) = 0 = u(0,t) = u(l,t).$$
Physically this is heat flow in a rod which has a chemical reaction in it such that the reaction produced inside the rod dies out over time.

(6) (4.3.5) Find the general solution of
$$u_{tt} = c^2 u_{xx} - a^2 u,$$
$$u(0,t) = u(l,t) = 0,$$
with arbitrary initial conditions. Physically, this is a model for a string vibrating in an elastic medium where the term $-a^2u$ represents the force of reaction of the medium on the string.

3.1.2. *To solve oneself.*

(1) (EO 25) Solve the problem:
$$u_{xx} + u_{yy} = y, \quad 0 < x < 2, \quad 0 < y < 1$$
$$u(x,0) = 0, \quad u(x,1) = 0$$
$$u(0,y) = y - y^3, \quad u(2,y) = 0.$$

(2) (EO 27) Solve the problem
$$u_{xx} + u_{yy} + 20u = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$
$$u(0,y) = u(1,y) = 0$$
$$u(x,0) = 0, \quad u(x,1) = x^2 - x.$$

(3) (4.4:1) Solve the equation
$$u_{xx} + u_{yy} = 0$$
inside the square $0 \le x, y \le l$, subject to the boundary conditions:
$$u(x,0) = u(0,y) = u(l,y) = 0, \quad u(x,l) = x(l-x).$$

(4) (EO 3) Expand the function $\cos(x)$ in a sine series on the interval $(0, \pi/2)$. Use the result to compute

$$\sum_{n \geq 1} \frac{n^2}{(4n^2 - 1)^2}.$$

(5) (4.2.2) Solve:

$$u_t = k u_{xx}, \quad u(x, 0) = f(x),$$
$$u(0, t) = C \neq 0, \quad u_x(l, t) = 0.$$

(6) (4.3.1) Show that the function

$$b_n(t) := \frac{1}{n\pi c} \int_0^t \sin \frac{n\pi c(t-s)}{l} \beta_n(s) ds$$

solves the differential equation:

$$b_n''(t) + \frac{n^2 \pi^2 c^2}{l^2} b_n(t) = \beta_n(t),$$

as well as the initial conditions $b_n(0) = b_n'(0) = 0$.

(7) (4.4.7) Solve the Dirichlet problem:

$$u_{xx} + u_{yy} = 0 \text{ in } S = \{(r, \theta) : 0 < r_0 \leq r \leq 1, \quad 0 \leq \theta \leq \beta\},$$
$$u(r_0, \theta) = u(1, \theta) = 0, \quad u(r, 0) = g(r), \quad u(r, \beta) = h(r).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).