Remember, though, that we're doing "algebras defined by generators and relations". The $X$ in $T(\Omega, X)$ is the set of generators, but what about the relations? These define a congruence on $\mathcal{C}(\Omega, X)$, but in order to sort out how, we need to study the congruences in more detail.

In the phrase "defined by generators and relations", the term 'relation' is really used rather informally, as a synonym of "law", "identity", "equality", and "axiom". Somewhat surprisingly, though, it ends up amounting to precisely a relation (on $T(\Omega, X)$) in the technical sense, i.e., a subset of $T(\Omega, X) \times T(\Omega, X)$.

**Def.** A relation/"law/identity/... $E \subseteq T(\Omega, X) \times T(\Omega, X)$ is said to hold in the quotient $T(\Omega, X)/\sim$ if $E = \sim$. More generally, $E$ holds in an $\Omega$-algebra $B$ (with respect to an $\Omega$-algebra homomorphism $\phi: T(\Omega, X) \to B$) if $E \subseteq \text{Ker} \phi$.

**Example.** Let $\Omega \ni m(\cdot, \cdot)$. Then commutativity of $m$ is the relation

$$\left\{ (m(s, t), m(t, s)) \mid s, t \in T(\Omega, X) \right\} =: E_{\text{com}}$$

and associativity is

$$\left\{ (m(m(t, s), r), m(t, s, r)) \mid r, s, t \in T(\Omega, X) \right\} =: E_{\text{ass}}.$$

Any epimorphism $\phi: T(\Omega, X) \to B$ for which $E_{\text{com}} \subseteq \text{Ker} \phi$ forces $m^B$ to be commutative.
Example. A unital C-algebra can be taken as having the signature 
\( \Omega = \{ p(\cdot), m(\cdot), 0, 1, \overline{3} \cup \mathbb{R} \} \), \( z \in \mathbb{C} \).

The q-commutation relation \( yx = qxy + 1 \) of a q-deformed
Heisenberg algebra is in this signature

\[
\left\{ \left( m(y,x), p\left( s_q(m(x,y)), 1 \right) \right) \right\} \in T(\Omega, 3, x, y) \times T(\Omega, 3, x, y) 
\]

not an infinite family of pairs, but just one pair.

One complication here is that none of these three relations
are congruence relations, but forming a quotient requires
a congruence. What to do? We pick the congruence corresponding
to the largest (and thus least restrictive) quotient, which turns
out to be the congruence which is smallest as a set. Obviously,

\[ (s,t) \in R_1 \Rightarrow (s,t) \in R_2 \text{ then } R_1 \subseteq R_2 \]

so \( R_1 \subseteq R_2 \) means everything for which \( R_2 \) holds will also be something
for which \( R_1 \) holds. The smaller (as a set) a congruence is,
the fewer things will it identify in the quotient. Furthermore...

Lemma. Let \( A \) be an \( \mathbb{C} \)-algebra and \( C \) a nonempty subset of
\( \mathbb{C} \)-algebra congruences on \( A \) (so \( C \subseteq 2^{\text{Con}(A)} \)). Then
\( Q = \bigcap_{R \in C} R \) is an \( \mathbb{C} \)-algebra congruence on \( A \).

Proof. Just check the conditions. \( \square \)
Construction. Let a signature $\Sigma$ and set of generators $X$ be given. Let $E \subseteq T(\Sigma, X) \times T(\Sigma, X)$ (the "relations") be arbitrary. Define

$$Q = \bigcap_{R \in E} R$$

and

$$A = T(\Sigma, X) / Q.$$

Then $A$ is called the free $\Sigma$-algebra on (the generators) $X$ satisfying (the relations/identities/laws) $E$, and may be denoted $\mathcal{L}(X | E)$ (often with the sets $X$ and $E$ expanded).

It has the property that if $\phi : T(\Sigma, X) \to B$ is any $\Sigma$-algebra homomorphism satisfying $E$, then $\phi = \Theta \circ \text{eval}(Q)$ where $\Theta : A \to B$ is an $\Sigma$-algebra homomorphism.

Proof. $T(\Sigma, X) / T(\Sigma, X)$ is a congruence on $T(\Sigma, X)$ (one that identifies everything, so the quotient would have just one element), meaning the set of congruences intersected above is nonempty. Hence $Q$ is a congruence by the lemma, and $A$ is well-defined.

For the other claim, we know that $E \subseteq \ker \phi$, so $\ker \phi$ is one of the congruences in the intersection, and thus $Q \subseteq \ker \phi$. The claim $\phi$ decomposes as $\Theta \circ \text{eval}(Q)$ then amounts to

$$\Theta([t]_Q) = \phi(t) \text{ for all } t \in T(\Sigma, X)$$

which clearly defines $\Theta$ on all of $A$. This definition is consistent since

$$[t]_Q = [t]_Q \iff (\sigma, t) \in Q \iff (\sigma, t) \in \ker \phi \iff \phi(\sigma) = \phi(t).$$

That $\Theta$ is a homomorphism is checked as in the proof of the First Isomorphism Theorem.
What kind of axioms can be turned into "relations"?

Above examples demonstrate that we're OK with:
- Equalities of specific elements, e.g. \( ab = 2 \).
- All-quantified equalities, e.g. \( \forall x \forall y : \Phi(x,y) = \Psi(y,x) \).

What about existence of a (left) inverse, e.g. \( \forall x \exists y : y \cdot x = 1 \)? That's fine too, thanks to the technique of Skolemisation (after Norwegian logician Thoralf Skolem): For every \( \exists \) introduce a new function symbol in \( \mathcal{O} \) taking all other \( \forall \)-quantified variables as operands, and use the value instead of the \( \exists \)-quantified variable. Thus \( \forall x \exists y : y \cdot x = 1 \) becomes \( \forall x : i(x) \cdot x = 1 \). Skolemisation can deal with an arbitrarily deep nesting of \( \forall \) and \( \exists \).

It does not matter if the existence is non-unique (e.g. left inverse when there is no right inverse), although it can make the free algebra a more complicated object than one first expected. \( i(x) \) might become some sort of idealised "generic" inverse of \( x \), distinct from each suspected candidate, but possible to map to any of them using a suitable endomorphism.

So what about the inverse of \( 0 \) in a ring? That turns out to be \( 0 \), but in an uninteresting way. Allowing the axiom \( \forall x : i(x) \cdot x = 1 \) (to those of a ring adds in particular \( i(1) \cdot 0 = 1 \) from which follows \( 0 = i(0) \cdot 0 = 1 \), i.e. any congruence \( Q \) subsuming \( i(0) \cdot 0 \equiv 1 \) also satisfies \((0,1) \in Q\), so in fact \( Q = \{ (0, x) \times \{ (2, x) \} \} \) and the quotient only has one element: \( i(0) = 0 \) because everything is \( 0 \).
How about the multiplicative inverse in a field? Here the axiom structure is

\[ \forall x: (x \neq 0 \Rightarrow \exists y: y \cdot x = 1) \]

and what wrecks things is not so much the \( \exists y: y \cdot x = 1 \) as the \( \forall x \neq 0 \Rightarrow \); the set of congruences satisfying such an implication need not be closed under intersection, so you can't transition to a free \( \mathbb{Q} \)-algebra having that property. More generally, fields are too rigid to fit into the universal algebra framework (except as "scalars").

What one could do is add an axiom \( \forall x, x \cdot i(x) \cdot x = x \) and have that hold for fields, but the corresponding free algebra wouldn't be a field.

Other things that won't translate as "relations" are axioms with \( \lor \) in them, e.g. the above is equivalent to \( \forall x: (x = 0 \lor \exists y: y \cdot x = 1) \). And is on the other hand not a problem: just pour in each constraint into the relation \( E \).