FOURIER ANALYSIS & METHODS 2020.02.04

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. The best approximation theorem

The Fourier series of f an element of a Hilbert space, H, with respect to an orthonormal set $\{\phi_n\}$ is

$$\sum_n \hat{f}_n \phi_n,$$

where

$$\hat{f}_n = \langle f, \phi_n \rangle, \text{ and the set } \{\phi_n\} \text{ is orthonormal, meaning } \langle \phi_n, \phi_m \rangle = \begin{cases} 1 & n = m \\ 0 & n \neq m. \end{cases}$$

The Fourier series is actually *equal* to f if and only if the orthonormal set is in fact an orthonormal basis. In any case, even though the Fourier series might not be equal to f, it is the *best* approximation to f in the following sense.

Theorem 1 (Best Approximation)**.** *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal set in a Hilbert space, H. If $f \in H$, and*

$$\sum_{n\in\mathbb{N}} c_n \phi_n \in H,$$

then

$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n \rangle \phi_n|| \leq ||f - \sum_{n\in\mathbb{N}} c_n \phi_n||, \quad \forall \{c_n\}_{n\in\mathbb{N}} \in \ell^2,$$

and equality holds $\iff$ $c_n = \langle f, \phi_n \rangle$ is true $\forall n \in \mathbb{N}$.

Proof: We make a few definitions: let

$$g := \sum \widehat{f_n} \phi_n, \quad \widehat{f_n} = \langle f, \phi_n \rangle,$$

and

$$\varphi := \sum c_n \phi_n.$$

Idea: write

$$||f - \varphi||^2 = ||f - g + g - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 + 2\Re \langle f - g, g - \varphi \rangle.$$

Idea: show that

$$\langle f - g, g - \varphi \rangle = 0.$$

Just write it out (stay calm and carry on):

$$\langle f, g\rangle - \langle f, \varphi\rangle - \langle g, g\rangle + \langle g, \varphi\rangle$$

$$= \sum \overline{\widehat{f_n}} \langle f, \phi_n\rangle - \sum \overline{c_n} \langle f, \phi_n\rangle - \sum \widehat{f_n} \langle \phi_n, \sum \widehat{f_m} \phi_m\rangle + \sum \widehat{f_n} \langle \phi_n, \sum c_m \phi_m\rangle$$

$$= \sum |\widehat{f_n}|^2 - \sum \overline{c_n} \widehat{f_n} - \sum |\widehat{f_n}|^2 + \sum \widehat{f_n} \overline{c_n} = 0,$$

where above we have used the fact that ϕ_n are an orthonormal set. Then, we have

$$||f - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 \geq ||f - g||^2,$$

with equality iff

$$||g - \varphi||^2 = 0.$$

Let us now write out what this norm is, using the definitions of g and φ. By their definitions,

$$g - \varphi = \sum (\widehat{f_n} - c_n)\phi_n.$$

By the Pythagorean theorem, due to the fact that the ϕ_n are an orthonormal set, and hence multiplying them by the scalars, $\widehat{f_n} - c_n$, they remain orthogonal, we have

$$||g - \varphi||^2 = \sum \left| \widehat{f_n} - c_n \right|^2 .$$

This is a sum of non-negative terms. Hence, the sum is only zero if all of the terms in the sum are zero. The terms in the sum are all zero iff

$$\left| \widehat{f_n} - c_n \right| = 0 \forall n \iff c_n = \widehat{f_n} \forall n \in \mathbb{N}.$$

Corollary 2. *Assume that $\{\phi_n\}$ is an OS in a Hilbert space H. Then the best approximation to $f \in H$ of the form*

$$\sum_{n=1}^{N} c_n \phi_n$$

is given by taking

$$c_n = \frac{\langle f, \phi_n\rangle}{||\phi_n||^2}.$$

Exercise 1. *Prove this corollary using the best approximation theorem.*

1.1. **Application of the best approximation theorem.** The goal is to find the numbers $\{c_j\}_{j=0}^3$ so that

$$\int_{-\pi}^{\pi} |f - \sum_{j=0}^{3} c_j e^{ijx}|^2 dx$$

is minimized. Here,

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 \leq x \leq \pi \end{cases}$$

Since the functions e^{ijx} are orthogonal on $\mathcal{L}^2(-\pi,\pi)$ we can apply the best approximation theorem! It says that the best approximation is to set The best approximation theorem's corollary says that

$$c_j = \frac{\hat{f}_j}{||e^{ijx}||^2} = \frac{\langle f, \phi_j \rangle}{||\phi_j||^2}, \quad \phi_j(x) = e^{ijx}.$$

We therefore compute

$$c_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ijx}dx = \begin{cases} \frac{1}{2} & j = 0 \\ \frac{(-1)^j - 1}{-2\pi i j} & j = 1, 2, 3 \end{cases}$$

2. Spectral Theorem Motivation

Partial differential operators act on functions which are elements of certain Hilbert spaces, known as Sobolev spaces. For example, the operator

$$\Delta = -\partial_x^2$$

acts on the Hilbert space H^2. Don't worry about what it is precisely, because all that matters is that it is a Hilbert space. The operator Δ takes elements of the Hilbert space H^2 and sends them to the Hilbert space $\mathcal{L}^2$. It is a linear operator because

$$\partial_x^2(f(x) + g(x)) = f''(x) + g''(x) = \partial_x^2(f(x)) + \partial_x^2(g(x)).$$

Thinking of functions as vectors, then Δ is like a linear map that takes in vectors and spits out vectors. Just like linear maps on finite dimensional vector spaces, which can be represented by a matrix, a linear operator on a Hilbert space can be represented by a matrix. If it is a sufficiently "nice" operator, then there will exist an orthonormal basis of eigenfunctions with corresponding eigenvalues. Here it is useful to recall

Theorem 3 (Spectral Theorem for $\mathbb{C}^n$)**.** *Assume that A is a Hermitian matrix. Then there exists an orthonormal basis of $\mathbb{C}^n$ which consists of eigenvectors of A. Moreover, each of the eigenvalues is real.*

Proof: Remember what Hermitian means. It means that for any $u, v \in \mathbb{C}^n$, we have

$$\langle Au, v \rangle = \langle u, Av \rangle.$$

By the Fundamental Theorem of Algebra, the characteristic polynomial

$$p(x) := \det(A - xI)$$

factors over $\mathbb{C}$. The roots of p are $\{\lambda_k\}_{k=1}^n$. These are by definition the eigenvalues of A. First, we consider the case when A has zero as an eigenvalue. If this is the case, then we define

$$K_0 := \mathrm{Ker}(A) = \{u \in \mathbb{C}^n : Au = 0\}.$$

We note that all nonzero $u \in K_0$ are eigenvectors of A for the eigenvalue 0. Since K_0 is a k-dimensional subspace of $\mathbb{C}^n$, it has an ONB $\{v_1, \ldots, v_k\}$. If $k = n$, we are done. So, assume that $k < n$. Then we consider

$$K_0^{\perp} = \{u \in \mathbb{C}^n : \langle u, v \rangle = 0 \forall v \in K_0\}.$$

Note that if $u \in K_0^{\perp}$ then

$$\langle Au, v \rangle = \langle u, Av \rangle = 0 \quad \forall v \in K_0.$$

Hence $A : K_0^\perp \to K_0^\perp$. Moreover, if

$$u \in K_0^\perp, \quad Au = 0 \implies u \in K_0 \cap K_0^\perp \implies u = 0.$$

Hence A is bijective from $K_0^\perp$ to itself. Since A has eigenvalues $\{\lambda_j\}_{j=1}^n$, and 0 appears with multiplicity k, $\lambda_{k+1} \neq 0$. It has some non-zero eigenvector. Let's call it u. Since it is an eigenvector it is not zero, so we define

$$v_{k+1} := \frac{u}{||u||}.$$

Proceeding inductively, we define K_1 to be the span of the vectors $\{v_1, \ldots, v_{k+1}\}$. We look at A restricted to $K_1^\perp$. We note that A maps K_1 to itself because if

$$v = \sum_1^{k+1} c_j v_j \implies Av = \sum_1^{k+1} c_j A v_j = \sum_1^{k+1} c_j \lambda_j v_j \in K_1.$$

Similarly, if $w \in K_1^\perp$,

$$\langle Aw, v \rangle = \langle w, Av \rangle = 0 \forall v \in K_1.$$

So, A maps $K_1^\perp$ into itself. Since the kernel of A is in K_1, A is a surjective and injective map from $K_1^\perp$ into itself. We note that A restricted to $K_1^\perp$ satisfies the same hypotheses as A, in the sense that it is still Hermitian, and it has a characteristic polynomial of degree equal to the dimension of $K_1^\perp$ So, there is an eigenvalue λ_{k+2}, for A as a linear map from $K_1^\perp$ to itself. It has an eigenvector, which we may assume has unit length, contained in $K_1^\perp$. Call it v_{k+2}. Continue inductively until we reach in this way $\{v_1, \ldots, v_n\}$ to span $\mathbb{C}^n$.

Why are the eigenvalues all real? This follows from the fact that if λ is an eigenvalue with eigenvector u then

$$\langle Au, u \rangle = \lambda ||u||^2 = \langle u, Au \rangle = \overline{\lambda} ||u||^2.$$

Since u is an eigenvector it is not zero, so this forces $\lambda = \overline{\lambda}$.

2.1. **An example.** Let us do an example. On $[-\pi, \pi]$, the functions which satisfy

$$\Delta f = \lambda f, \quad f(-\pi) = f(\pi)$$

are

$$f(x) = f_n(x) = e^{inx}.$$

The corresponding

$$\lambda_n = n^2.$$

So, the eigenvalues of Δ with this particular boundary condition are n^2, and the corresponding eigenfunctions are $e^{\pm inx}$. We have proven that these are orthogonal. We note that for all f and g in $\mathcal{L}^2$ which satisfy $f(-\pi) = f(\pi)$, $g(-\pi) = g(\pi)$ and which are also (at least weakly) twice differentiable, we would also get $f'(-\pi) = f'(\pi)$ and similarly for g, so that

$$\langle \Delta f, g \rangle = \int_{-\pi}^{\pi} -f''(x)\overline{g(x)}dx = -f'(x)\overline{g(x)}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f'(x)\overline{g'(x)}dx$$

$$= -f'(x)\overline{g(x)}\Big|_{-\pi}^{\pi} + f(x)\overline{g'(x)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)\overline{g''(x)}dx.$$

Due to the boundary conditions, all that survives is

$$-\int_{-\pi}^{\pi} f(x)\overline{g''(x)}dx = \langle f, \Delta g\rangle.$$

So we see that

$$\langle \Delta f, g\rangle = \langle f, \Delta g\rangle.$$

This is just like the spectral theorem for Hermitian matrices! There is a similar spectral theorem here, a "grown-up linear algebra" theorem, called The Adult Spectral Theorem. This grown-up version of the spectral theorem says that, like a Hermitian matrix, the operator Δ also has an $\mathcal{L}^2$ orthonormal basis of eigenfunctions. Hence, by this Spectral Theorem, we will be able to conclude that the orthonormal *set,*

$$\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}},$$

is an ONB. To state the Adult Spectral Theorem, we need to introduce Regular Sturm-Liouville Problems (SLPs).

2.2. **Regular SLPs.** Let L be a linear, second order ordinary differential operator. So, we can write

$$L(f) = r(x)f''(x) + q(x)f'(x) + p(x)f(x).$$

Above, r, q, and p are specified REAL VALUED functions. As a simple example, take $r(x) = -1$, and $q(x) = p(x) = 0$. Then we have

$$L(f) = \Delta f = -f''(x).$$

We are working with functions defined on an interval $[a, b]$ which is a *finite* interval. So, the Hilbert space in which everything is happening is $\mathcal{L}^2$ on that interval. Like with matrices, we can think about the *adjoint* of the operator L. The adjoint by definition satisfies

$$\langle Lf, g\rangle = \langle f, L^* g\rangle,$$

where we are using L^* to denote the adjoint operator. Whatever it is. On the left side, we know what everything is, so we write it out by definition of the scalar product

$$\langle Lf, g\rangle = \int_a^b L(f)\overline{g(x)}dx = \int_a^b \left(r(x)f''(x) + q(x)f'(x) + p(x)f(x)\right)\overline{g(x)}dx.$$

Integrating by parts, we get

$$\begin{aligned} &= (r\bar{g})f'|_a^b - \int_a^b (r\bar{g})'f' + (qg)f|_a^b - \int_a^b (q\bar{g})'f + \int_a^b pf\bar{g} \\ &= (r\bar{g})f' + (q\bar{g})f|_a^b - \int_a^b \left[(r\bar{g})'f' + (q\bar{g})'f - pf\bar{g}\right]. \end{aligned}$$

We integrate by parts once more on the $(r\bar{g})'f'$ term to get

$$= (r\bar{g})f' - (r\bar{g})'f + (q\bar{g})f)|_a^b + \int_a^b (r\bar{g})''f - (q\bar{g})'f + fp\bar{g}.$$

So, if the boundary conditions are chosen to make the stuff evaluated from a to b (these are called the boundary terms in integration by parts) vanish, then we could define

$$L^* g = (rg)'' - (qg)' + pg,$$

since then

$$\langle Lf, g\rangle = \int_a^b (r\bar{g})'' f - (q\bar{g})' f + fp\bar{g} = \langle f, L^* g\rangle.$$

Here we use that r, q and p are real valued functions, so $\bar{r} = r$, $\bar{q} = q$, and $\bar{p} = p$. For the spectral theorem to work, we will want to have

$$L = L^*.$$

When this holds, we say that L *is formally self-adjoint.* So, we need

$$Lf = L^* f \iff rf'' + qf' + pf = (rf)'' - (qf)' + pf.$$

We write the things out:

$$rf''+qf'+pf = (rf'+r'f)'-qf'-q'f+pf \iff rf''+qf' = rf''+2r'f'+r''f-qf'-q'f$$

$$\iff qf' = 2r'f' + r''f - qf' - q'f \iff (2q - 2r')f' + (r'' - q')f = 0.$$

To ensure this holds for all f, we set the coefficient functions equal to zero:

$$2q - 2r' = 0 \implies q = r', \quad q' = r''.$$

Well, that just means that $q = r'$. So, we need L to be of the form

$$Lf = rf'' + r'f' + pf = (rf')' + pf.$$

The boundary terms should also vanish, so we want:

$$(r\bar{g})f' - (r\bar{g})'f + (q\bar{g})f)\big|_a^b = (r\bar{g})f' - (r\bar{g})'f + (r'\bar{g})f\big|_a^b = 0,$$

$$\iff r\bar{g}f' - r'\bar{g}f - r\bar{g}'f + r'\bar{g}f\big|_a^b = 0 \iff r\bar{g}f' - r\bar{g}'f\big|_a^b = 0$$

$$\iff r(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

So, it suffices to assume that we are working with functions f and g that satisfy

$$(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

Writing this out we get:

$$\bar{g}(b)f'(b) - \bar{g}'(b)f(b) - (\bar{g}(a)f'(a) - \bar{g}'(a)f(a)) = 0 \iff$$

$$\bar{g}(b)f'(b) - \bar{g}'(b)f(b) = \bar{g}(a)f'(a) - \bar{g}'(a)f(a).$$

This is how we get to the definition of a regular SLP on an interval $[a, b]$. It is specified by

(1) a formally self-adjoint operator

$$L(f) = (rf')' + pf,$$

where r and p are real valued, r, r', and p are continuous, and $r > 0$ on $[a, b]$.

(2) self-adjoint boundary conditions:

$$B_i(f) = \alpha_i f(a) + \alpha_i' f'(a) + \beta_i f(b) + \beta_i' f'(b) = 0, \quad i = 1, 2.$$

The self adjoint condition further requires that the coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i'$ are such that for all f and g which satisfy these conditions

$$r(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

(3) a positive, continuous function w on $[a, b]$.

The SLP is to find all solutions to the BVP

$$L(f) + \lambda w f = 0, \quad B_i(f) = 0, \quad i = 1, 2.$$

The eigenvalues are all numbers λ for which there exists a corresponding non-zero eigenfunction f so that together they satisfy the above equation, and f satisfies the boundary condition.

We then have a miraculous fact.

Theorem 4 (Adult Spectral Theorem)**.** *For every regular Sturm-Liouville problem as above, there is an orthonormal basis of L^2_w consisting of eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}}$ with eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$. We have*

$$\lim_{n\to\infty} \lambda_n = \infty.$$

Here, L^2_w is the weighted Hilbert space consisting of (the almost everywhere-equivalence classes of measurable) functions on the interval $[a, b]$ which satisfy

$$\int_a^b |f(x)|^2 w(x) dx < \infty,$$

and the scalar product is

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)} w(x) dx.$$

We are not equipped to prove this fact. You can rest assured however that it is done through the techniques of functional analysis and bears similarity to the proof of the spectral theorem for finite dimensional vector spaces. As a corollary to this theorem we obtain

Theorem 5. *The functions*

$$\left\{e^{inx}\right\}_{n\in\mathbb{Z}}$$

are an orthogonal basis for the Hilbert space $\mathcal{L}^2(-\pi, \pi)$.

Proof: These functions satisfy a regular SLP. This SLP is to find all constants λ and functions f such that

$$f'' + \lambda f = 0,$$

and f is 2π periodic. The operator L is just the operator

$$L(f) = f''.$$

The function $r = 1$, $p = 0$, and the weight is just 1. The boundary conditions are thus:

$$f(-\pi) - f(\pi) = 0, \quad f'(-\pi) - f'(\pi) = 0.$$

We can check that this is 'self-adjoint' by plugging it into the required condition. Assume that some totally arbitrary f and g satisfy this condition, so that $g(-\pi) - g(\pi) = 0$ also. Then

$$(\bar{g}f' - \bar{g}'f)|_{-\pi}^{\pi} = \bar{g}(\pi)f'(\pi) - \bar{g}'(\pi)f(\pi) - \bar{g}(-\pi)f'(-\pi) + \bar{g}'(-\pi)f(-\pi) = 0.$$

By our ODE theory, we can already say that all solutions (up to constant factors) to this problem are

$$f_n(x) = e^{inx}, \quad \lambda_n = n^2\pi^2.$$

Now, by the Adult Spectral Theorem, we know that these are an orthogonal basis (they can be normalized if we so desire).

2.3. Exercises for the week: Hints. Those exercises from folland [1] which one should solve are:

(1) (3.3.1) Show that if $\{f_n\}_{n \geq 1}$ are elements of a Hilbert space, H, and we have for some $f \in H$ that
$$\lim_{n \to \infty} f_n = f,$$
then for all $g \in H$ we have
$$\lim_{n \to \infty} \langle f_n, g \rangle = \langle f, g \rangle.$$
Hint: Apply the Cauchy-Schwarz inequality to $\langle f_n - f, g \rangle$.

(2) (3.3.2) Show that for all f, g in a Hilbert space one has
$$|||f|| - ||g||| \leq ||f - g||.$$
Hint: First show that for any real numbers a and b,
$$|a - b|^2 = a^2 - 2ab + b^2.$$
Next, apply this fact with $a = ||f||$ and $b = ||g||$ to show that
$$|||f|| - ||g||| = ||f||^2 - 2||f||||g|| + ||g||^2.$$
Compare this to
$$||f - g||^2 = ||f||^2 - 2\Re\langle f, g \rangle + ||g||^2.$$

(3) (3.3.10.d) Use Parseval's equation to compute
$$\sum_{n \geq 1} \frac{\sin^2(na)}{n^4}.$$
Hint: The Fourier series of
$$f(x) := \begin{cases} x & -a < x < a \\ a\frac{\pi - x}{\pi - a} & a < \theta < \pi \\ a\frac{\pi + x}{a - \pi} & -\pi < x < -a \end{cases}$$
where implicitly we are assuming $0 < a < \pi$ is
$$\frac{2}{\pi - a} \sum_{n \geq 1} \frac{\sin(na)}{n^2} \sin(nx)$$

(4) (3.4.1) Show that $\{e^{2\pi i(mx+ny)}\}_{n,m \in \mathbb{Z}}$ is an orthogonal set in $\mathcal{L}^2(R)$ where R is any square whose sides have length one and are parallel to the coordinate axes. Hint: Compute the integral
$$\int_{x=a}^{a+1} \int_{y=b}^{b+1} e^{2\pi i(mx+ny)} e^{-2\pi i(kx+\ell y)} dx dy, \quad m, n, k, \ell \in \mathbb{Z}.$$

(5) (3.4.6) Find an example of a sequence $\{f_n\}$ in $\mathcal{L}^2(0, \infty)$ such that $f_n(x) \to 0$ uniformly for all $x > 0$ but $f_n \not\to 0$ in the $\mathcal{L}^2$ norm. Hint: Oh this is a fun sort of challenge problem... Here is a little bit of idea. The function $\frac{1}{\sqrt{n^2+x}}$ is not in $\mathcal{L}^2(0, \infty)$. How about using this function as an idea, define functions $f_n(x)$ which are say defined in some way for $x \in [0, n]$ and make

them zero for all $x > n$. Get them to decrease uniformly to zero for all x, but get their $\mathcal{L}^2$ norms to be increasing... Play around with it!

(6) (3.5.7) Find all solutions f on $[0, 1]$ and all corresponding λ to the equation:

$$f'' + \lambda f = 0, \quad f(0) = 0, \quad f'(1) = -f(1).$$

Hint: As we have computed before, consider three cases, $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$. Use the boundary conditions to solve for all the possible f.

(7) (4.2.3) Let $f(x)$ be the initial temperature at the point x in a rod of length ℓ, mathematicized as the interval $[0, \ell]$. Assume that heat is supplied at a constant rate at the right end, in particular $u_x(\ell, t) = A$ for a constant value A, and that the left end is held at the constant temperature 0, so that $u(0, t) = 0$. Find a series expansion for the temperature $u(x, t)$ such that the initial temperature is given by $f(x)$. Hint: Divide and conquer. First find a so-called steady state solution, that is find a function $g(x)$ which does not depend on t which satisfies

$$(\partial_t - \partial_{xx})g = 0, \quad g(0) = 0, \quad g'(\ell) = A.$$

Now, since g does not depend on t, when you apply the heat operator you just get

$$-g''(x) = 0, \quad g(0) = 0, \quad g'(\ell) = A.$$

Find g which solves this. Now, look for a solution u which satisfies

$$u_t - u_{xx} = 0, \quad u(0, t) = u_x(\ell, t) = 0, \quad u(x, 0) = f(x) - g(x).$$

You can use the methods from last week, separation of variables, superposition (since everything including the BCs are homogeneous), and Fourier series (Hilbert spaces!) to solve for u. The full solution will then be

$$u(x, t) + g(x).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).