Chalmers/GU
Mathematics

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	18–04–05
Time:	8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Quanjiang Yu, tel. 0764-147839
Result announced:	18–05–03 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$
\begin{aligned}
\text{maximize} \quad & z = x_1 + 2x_2, \\
\text{subject to} \quad & x_1 + x_2 \geq -1, \\
& x_1 - x_2 \geq 1, \\
& x_1, \quad x_2 \geq 0.
\end{aligned}
$$

(2p) a) Solve the problem using phase I and phase II of the simplex method.

Aid: You may utilize the identity

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
$$

(1p) b) If an optimal solution exists, then use your calculations to decide whether it is unique or not. If the problem is unbounded, then use your calculations to specify a direction of unboundedness of the objective value.

(3p) Question 2

(gradient projection)

The gradient projection algorithm is a generalization of the steepest descent method to problems defined over convex sets. Given a point $\boldsymbol{x}_k$ the next point is obtained according to $\boldsymbol{x}_{k+1} = \text{Proj}_X[\boldsymbol{x}_k - \alpha_k \nabla f(\boldsymbol{x}_k)]$, where X is the convex set over which we minimize, $\alpha_k > 0$ is the step length, and $\text{Proj}_X(\boldsymbol{y}) := \text{argmin}_{x \in X} ||\boldsymbol{x} - \boldsymbol{y}||$ (i.e., the closest point in X to $\boldsymbol{y}$). Note that if $X = \mathbb{R}$ then the method reduces to the method of steepest descent.

Consider the optimization problem to

$$
\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := x_1^3 + 2x_2^2 + 2x_1x_2 - 2x_1, \\
\text{subject to} \quad & 0 \leq x_1 \leq 1, \\
& 0 \leq x_2 \leq 2.
\end{aligned}
$$

Start at the point $\boldsymbol{x}_0 = (0, 2)^{\mathrm{T}}$ and perform one iteration of the gradient projection algorithm using step length $\alpha_k = 1/8$. Note that the special form of the feasible region X makes the projection very easy! Is the point obtained a global/local optimum? Motivate why/why not!

(3p) Question 3

(optimality conditions for special feasible sets)

Consider the problem of minimizing the function $f(\boldsymbol{x}) := \sum_{j=1,\dots,n} f_j(x_j)$ over a set of the form $S := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \sum_{j=1}^n x_j = b;\ x_j \geq 0, \forall j \,\}$. We assume that f is in C^1 on S, and of course that $b > 0$, such that S is non-empty.

This problem is often referred to as the *resource allocation problem*, since it entails allocating fractions of the resource b to "activity levels" x_j in an optimal manner, considering the minimization of the "cost function" f, and the available resources, represented by the value of b.

Utilize the optimality conditions for differentiable optimization over closed, convex sets to establish that any stationary point $\boldsymbol{x}^*$ must satisfy the conditions that for some value $\mu^* \in \mathbb{R}$ it holds that $f_j'(x_j^*) = \mu^*$, for all j with $x_j^* > 0$, while $f_j'(x_j^*) \geq \mu^*$, for all j with $x_j^* = 0$.

Question 4

(Karush-Kuhn-Tucker)

Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := 2x_1 - x_1^2, \\ \text{subject to} \quad & x_1^2 + x_2^2 \geq 25, \\ & x_1 \leq 4, \\ & x_2 \leq 4, \\ & x_1, x_2 \geq 0. \end{aligned}$$

(2p) a) State the KKT-conditions for the problem and check whether they are necessary or not, sufficient or not.

(1p) b) Find all KKT-points. Are the KKT points optimal? Motivate!

(3p) Question 5

(modelling)

The *set covering problem* is a classical question in combinatorics, computer science and complexity theory. Given a set of elements $\mathcal{U} = \{1, 2, ..., n\}$ (called the universe) and a collection $\mathcal{S}$ of m sets whose union equals the universe, the *set cover problem* is the problem to identify the smallest sub-collection of $\mathcal{S}$ whose union equals the universe.

For example, consider the universe $\mathcal{U} = \{1, 2, 3, 4, 5\}$ and the collection of sets $\mathcal{S} = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$. Clearly the union of $\mathcal{S}$ is $\mathcal{U}$. However, we can cover all of the elements with the following, smaller number of sets: $\{\{1, 2, 3\}, \{4, 5\}\}$. This is also the smallest sub-collection whose union is $\mathcal{U}$.

A generalization of this problem is the *weighted set covering problem* where each set in $\mathcal{S}$ has a cost associated with it. The objective in the *weighted set covering problem* is to find a sub-collection of $\mathcal{S}$ whose union equals the universe, and so that the sum of the costs of the sets in the sub-collection is minimized.

Formulate an integer linear program (a linear objective function, linear constraints, and integrality restrictions on the variables) which models the weighted set covering problem.

Question 6

(true or false)

The below three claims should be assessed. Are they true or false, or is it impossible to say? Provide an answer, together with a short, but complete, motivation.

(1p) a) Suppose the function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a vector $\boldsymbol{x} \in \mathbb{R}^n$.

Claim: for the vector $\boldsymbol{p} \in \mathbb{R}^n$ to be a descent direction with respect to f at $\boldsymbol{x}$ it is necessary that $\nabla f(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{p} < 0$.

(1p) b) Suppose you attack the problem of minimizing the twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ by means of Newton's method, using an exact line search. Suppose the iterate is $\boldsymbol{x}^t$, and that the result of iteration t is the next iterate $\boldsymbol{x}^{t+1}$.

Claim: $\nabla f(\boldsymbol{x}^{t+1})^{\mathrm{T}}(\boldsymbol{x}^{t+1} - \boldsymbol{x}^t) = 0$ holds.

(1p) c) *Claim:* A line segment in $\mathbb{R}^n$ is not a polyhedron.

(3p) Question 7

(Farkas' lemma)

Farkas' Lemma can be states as follows:

Let $\boldsymbol{A}$ be any $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ vector. Then exactly one of the two systems

$$\boldsymbol{A}x = \boldsymbol{b},$$
$$\boldsymbol{x} \geq \boldsymbol{0}^n,$$

and

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{0}^m,$$
$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} > 0,$$

has a feasible solution, and the other system is inconsistent.

Prove Farkas' Lemma.