LECTURE 12: INTEGER LINEAR OPTIMIZATION

PROBLEM:

$$\boxed{\begin{aligned} \min \quad & \mathbb{c}^T \mathbb{x} \\ \text{s.t.} \quad & A\mathbb{x} \le \mathbb{b} \\ & \mathbb{x} \in \mathbb{Z}^n \end{aligned}} \qquad \text{(ILP)}$$

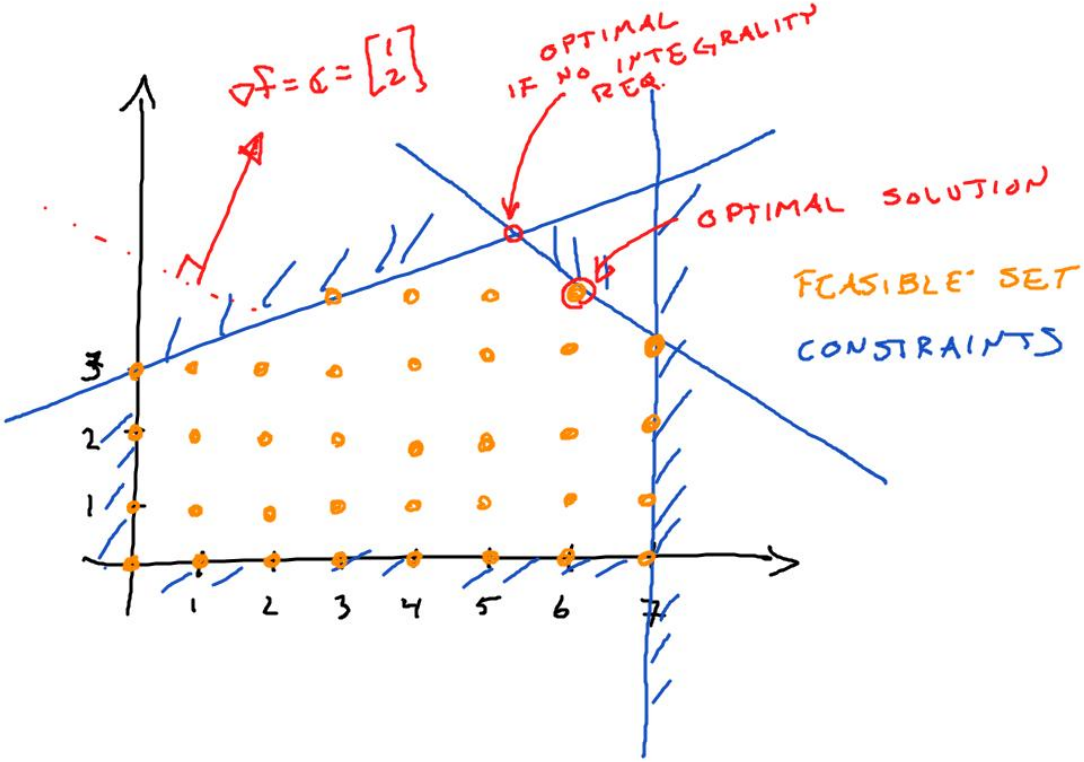
$$\begin{aligned} \min \quad & \mathbb{c}^T \mathbb{x} \\ \text{s.t.} \quad & A\mathbb{x} \le \mathbb{b} \\ & \mathbb{x} \in \{0,1\}^n \end{aligned} \qquad \text{(BP)}$$

$$\max \quad z_{IP} = x_1 + 2x_2$$

$$\text{s.t.} \quad \begin{aligned} x_1 + x_2 &\le 10 \\ -x_1 + 3x_2 &\le 9 \\ x_1 \quad &\le 7 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 & \text{ INTEGER} \end{aligned}$$

$$x^*_{IP} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \quad z^*_{IP} = 14$$

$$x^*_{LP} = \begin{bmatrix} 21/4 \\ 19/4 \end{bmatrix}, \quad z^*_{LP} = 14\tfrac{3}{4}$$



NOTE: $z^*_{LP} > z^*_{IP}$ [WHY? RELAXATION]

WHEN ARE ILPs USEFUL?

- PRODUCTS OR RAW MATERIAL ARE INDIVISIBLE
- LOGICAL CONSTRAINTS: E.G. "IF A THEN B", "A OR B"
- FIXED COSTS
- COMBINATORICS
- ON/OFF DECISIONS

NOTE: $x \in \{0,1\} \iff x(x-1)=0$

LOGICAL CONSTRAINTS

0-1 VARIABLES (BINARY) ARE VERY USEFUL IN MODELLING.

EX.

- "IF x THEN y": $\quad x \leq y$
- "XOR": $\quad x + y = 1$
- "OR": $\quad x + y \geq 1$
- "EXACTLY ONE OUT OF n MUST BE TRUE": $\quad x_1 + \dots + x_n = 1$
- "AT LEAST m OUT OF n MUST BE TRUE": $\quad x_1 + \dots x_n \geq m$
- ....

DISJOINT FEASIBLE SETS:

- "EITER $0 \le x \le 1$ OR $5 \le x \le 8$"

INTRODUCE A BINARY VARIABLE $y \in \{0,1\}$

$$\Rightarrow \quad \begin{aligned} & x \geq 0 \\ & x \leq 8 \\ & x \leq 1 + 7y \\ & x \geq 5y \\ & y \in \{0,1\} \end{aligned} \quad \Rightarrow$$

IF $y = 0 \Rightarrow \left\{ \begin{aligned} & x \geq 0 \\ & x \leq 8 \\ & x \leq 1 \\ & x \geq 0 \end{aligned} \right. \Rightarrow 0 \le x \le 1$

IF $y = 1 \Rightarrow \begin{aligned} & x \geq 0 \\ & x \leq 8 \\ & x \leq 8 \\ & x \geq 5 \end{aligned} \Rightarrow 5 \le x \le 8$

"VARIABLE x CAN ONLY TAKE VALUES 2, 45, 78 OR 107"

$$x = 2y_1 + 45y_2 + 78y_3 + 107y_4$$

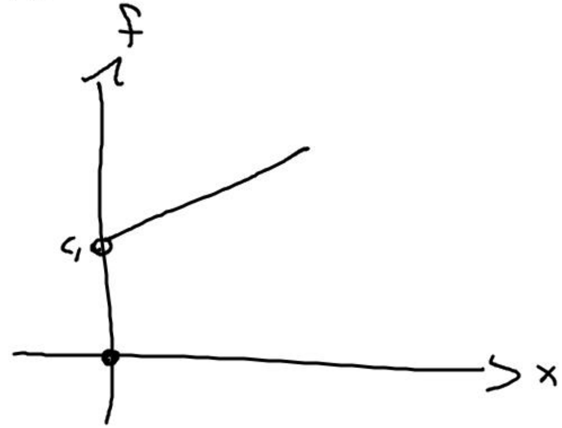
$$y_1 + y_2 + y_3 + y_4 = 1$$

$$y_1, y_2, y_3, y_4 \in \{0,1\}$$

- WE WANT TO MINIMIZE AN OBJ. FUNC. WITH FIXED COST:

$$f(x) = \begin{cases} 0 & \text{IF } x = 0 \\ c_1 + c_2 x & \text{IF } x > 0 \end{cases}$$

WHERE $c_1 > 0$



- INTRODUCE BINARY VARIABLE $y \in \{0,1\}$

$$f(x,y) = c_1 y + c_2 x$$

$x \geq 0$

$x \leq My$ WHERE $M > 0$ IS SOME LARGE NUMBER.

$y \in \{0,1\}$

"BIG-M CONSTRAINTS"

IS INTEGER OPTIMIZATION DIFFICULT?

- IN SOME SENSE, NO! E.G. FOR BINARY PROGRAMS, WE COULD JUST ENUMERATE ALL POSSIBLE SOLUTIONS. 2^n POSSIBLE SOLUTIONS

  THIS GROWS FAST

- IN GENERAL, INTEGER PROGRAMS ARE NP-HARD. MEANING THAT THEY CANNOT BE SOLVED IN POLYNOMIAL TIME

  "PROBABLY"

  n^k

  SLOW

SOLUTION METHODS FOR ILPs

- GENERAL SOLUTION METHODS [SLOW IN GENERAL]
  - BRANCH & BOUND METHOD *
  - CUTTING PLANE METHOD *
  - DYNAMIC PROGRAMMING
  - ALGEBRAIC METHODS.
- EXACT METHODS FOR SPECIFIC PROBLEM TYPES [FAST]
  - SHORTEST PATH PROBLEM
  - MINIMUM CUT
  - MINIMUM SPANNING TREE
  - ASSIGNMENT PROBLEMS

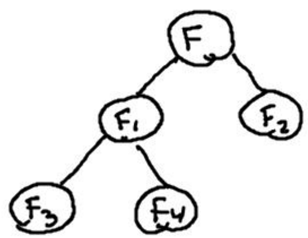
BRANCH & BOUND METHOD

- IDEA IS TO DIVIDE FEASIBLE SET F INTO $F_1, F_2, \dots, F_k$ WHERE $\bigcup F_i = F$.

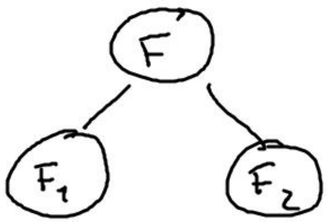
- INSTEAD OF SOLVING
$$\min c^T x \quad \text{s.t. } x \in F$$

  WE TRY TO SOLVE
$$\min c^T x \quad \text{s.t. } x \in F_i$$
  FOR ALL $i = 1, \dots, k$.

- DO THIS RECURSIVELY:



- ~~Do~~ $(P_i): \quad \min \ c^T x \quad \text{s.t.} \ x \in F_i$



- WE CAN STOP DIVIDING WHEN:

  a) IF WE MANAGE TO SOLVE P_i ← IF WE GET LUCKY WHEN REMOVING INTEGRALITY REQUIREMENTS

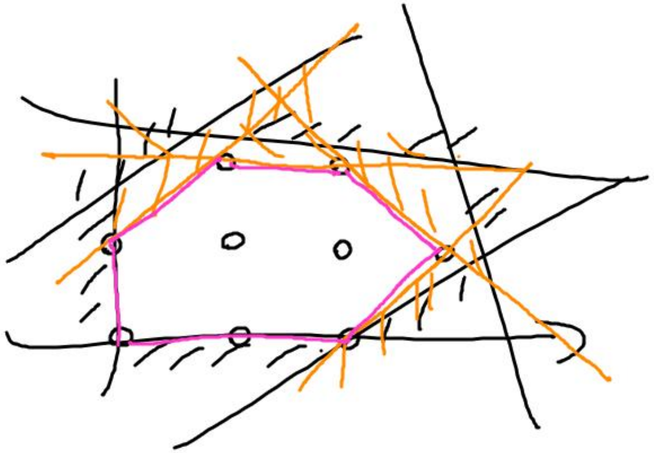
  b) IF $F_i = \emptyset$ ← IF THE LP-RELAXATION IS INFEASIBLE

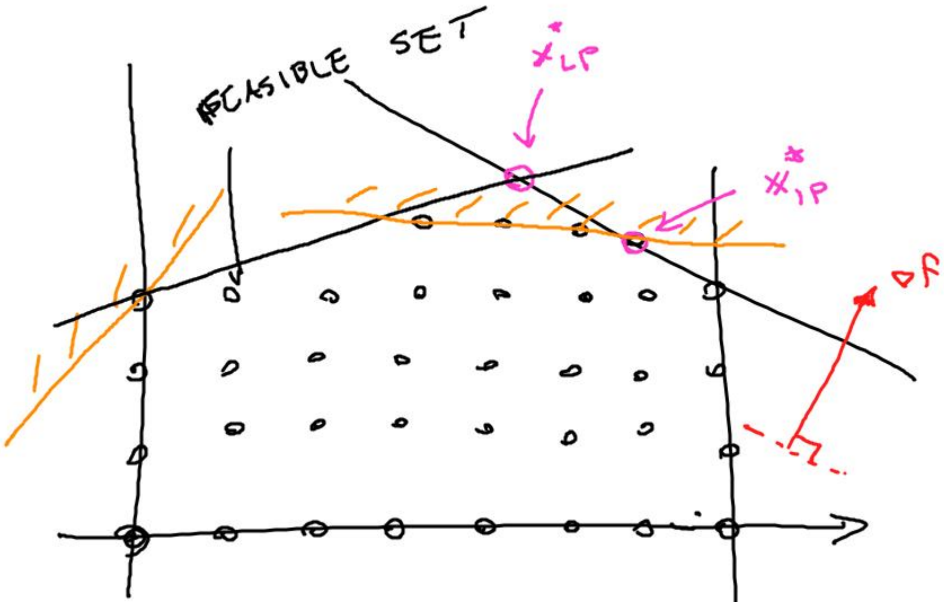
  c) BOUNDING: IF WE MANAGE TO FIND A LOWER BOUND $b(P_i)$ THAT IS HIGHER THAN OUR BEST FEASIBLE SOLUTION SO FAR. i.e. IF $b(P_i) \geq z_{BEST}$

  WE GET $b(P_i)$ BY SOLVING THE LP-RELAXATION

CUTTING PLANE

IDEA: CAN WE MAKE EXTREME POINTS INTEGER BY ADDING MORE LINEAR CONSTRAINTS?





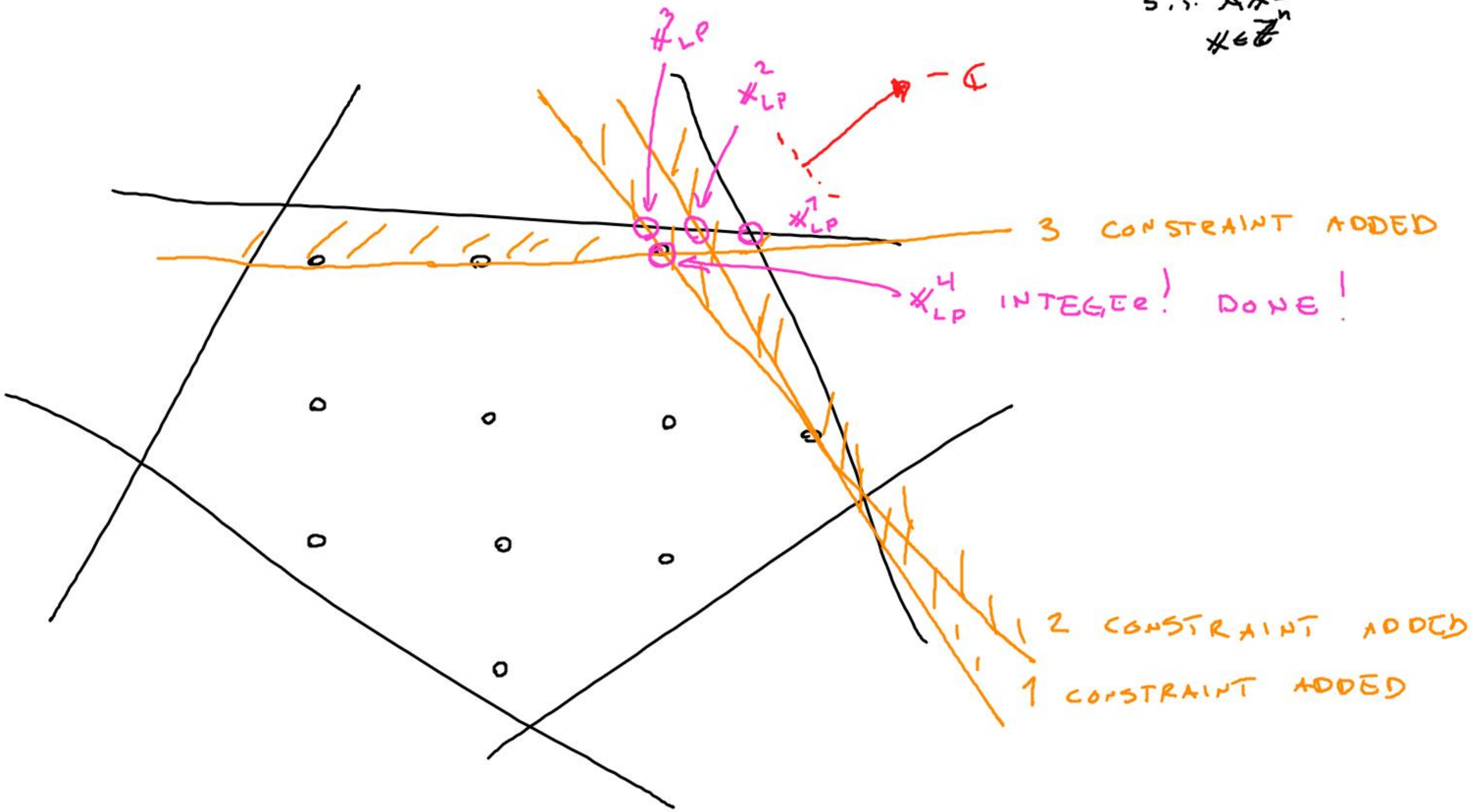
METHOD:

1: SOLVE LP-RELAXATION (REMOVE INT. REQ)

2: IF SOLUTION IS INTEGER, OK!
IF NOT, ADD A CONSTRAINT THAT CUTS AWAY THAT SOLUTION BUT NO INT. POINTS.

3: GO TO 1

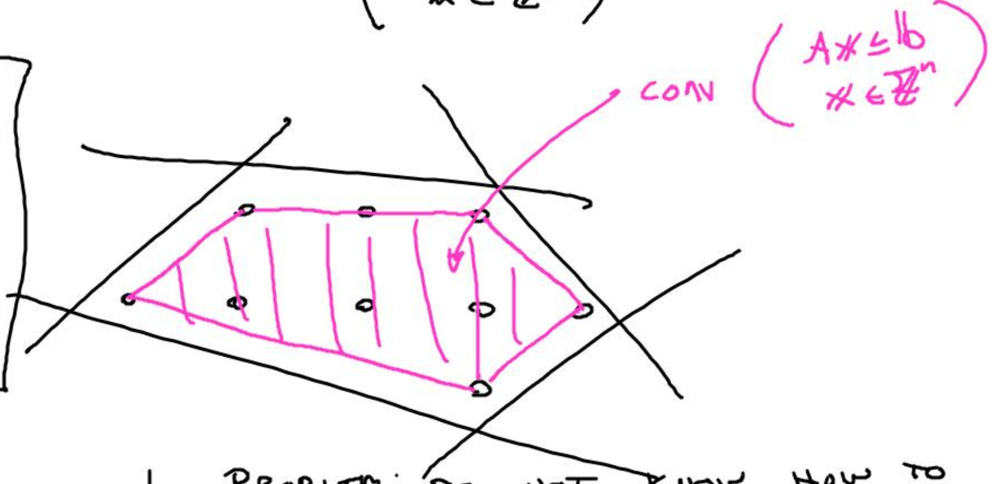
$$\min c^T x \quad \text{s.t. } Ax \le b, \quad x \in \mathbb{Z}^n$$



THERY BEHIND CUTTING PLANE METHOD

$$(IP)\quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \le b \\ & x \in \mathbb{Z}^n \end{array} \iff (R)\quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & x \in \operatorname{conv}\begin{pmatrix} Ax \le b \\ x \in \mathbb{Z}^n \end{pmatrix} \end{array}$$

THM: LET A BE A RATIONA MATRIX AND b A RATIONA VECTOR. THEN $\operatorname{conv}\begin{pmatrix} Ax \le b \\ x \in \mathbb{Z}^n \end{pmatrix}$ IS A POLYHEDRON



$\Rightarrow$ (R) IS A LINEAR PROGRAM!

PROBLEM: DO NOT KNOW HOW TO DESCRIBE $\operatorname{conv}(Ax \le b, x \in \mathbb{Z}^n)$ EASILY.