LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

Exam in Optimization for Learning

Test exam

Points and grading

All answers must include a clear motivation. Answers should be given in English. The total number of points is 20. The maximum number of points is specified for each subproblem. Preliminary grading scales:

Grade 3:	12 points on the exam
4:	17 points on exam plus extra-credit handin
5:	22 points on exam plus extra-credit handin

Accepted aid

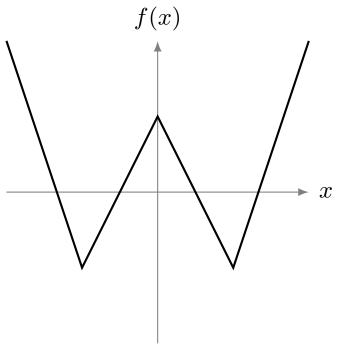
Authorized *Cheat Sheet*.

Results

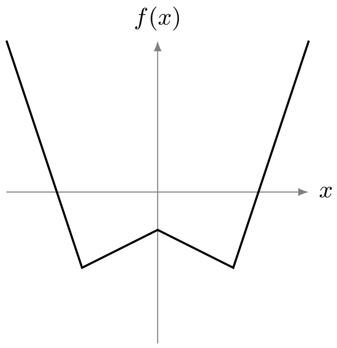
Solutions will be posted on the course webpage, and results will be registered in LADOK. Date and location for display of corrected exams will be posted on the course webpage.

1. Are the following sets C convex? You may, where applicable and without proving it, use that $\|x\|_\infty := \max_i(|x_i|)$ is a convex function and that $\{x \in \mathbb{R}^n : x \geq 0\}$ is a convex set.

a. Let $C = \{x \in \mathbb{R} : f(x) \leq 0\}$ where f is given below: (1 p)



b. $C = \{x \in \mathbb{R} : f(x) \leq 0\}$ where f is given below: (1 p)



c. $C = \{x \in \mathbb{R}^n : x \geq 0 \text{ and } \|x\|_\infty \leq 1\}$. (1 p)

d. $C = \{x \in \mathbb{R}^n : x \geq 0 \text{ or } \|x\|_\infty \leq 1\}$. (1 p)

e. $C = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \|x\|_\infty \leq t\}$. (1 p)

2. Are the following functions f convex? Where applicable, you may use convexity preserving operations or graphical arguments.

a. $f(x) = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (1 p)

b. $f(x) = e^{\|x\|_1}$ where $x \in \mathbb{R}^n$ (1 p)

c. $f(x) = \left|e^{\|x\|_1}\right|$ where $x \in \mathbb{R}^n$ (1 p)

d. $f(x) = \max(f(x), g(x))$ where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ are both convex. (1 p)

e. $f(x) = \min(x^2, x)$ where $x \in \mathbb{R}$ (1 p)

f. $f(x) = \max(x^2, x, -x^2)$ where $x \in \mathbb{R}$ (1 p)

g. $f(x) = \max(x^2, x, 1 - x^2)$ where $x \in \mathbb{R}$ (1 p)

3. Consider the following problem

$$\underset{u}{\text{minimize}}\, \max(h_1^T u, h_2^T u) + \tfrac{1}{2}\|u\|_2^2.$$

Show that u solves this if and only if (u, t) solves

$$\begin{array}{ll} \underset{u,t}{\text{minimize}} & t + \tfrac{1}{2}\|u\|_2^2 \\ \text{subject to} & h_1^T u \leq t \\ & h_2^T u \leq t \end{array}$$

(1 p)

4. Consider the following problem (that satisfies constraint qualification):

$$\underset{x}{\text{minimize}}\, \underbrace{\|Ax - b\|_1}_{f(Ax)} + \underbrace{\sum_{i=1}^{n} g_i(x_i)}_{g(x)}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, all $g_i : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ are the same and satisfy

$$g_i(x_i) = \begin{cases} \frac{1}{2}x_i^2 & \text{if } x_i \in [-1, 1] \\ \infty & \text{else} \end{cases}$$

and $f(y) = \|y - b\|_1$ with dual problem

$$\underset{\mu}{\text{minimize}}\, f^*(\mu) + g^*(-A^T \mu).$$

a. Compute the conjugate f^*. (1 p)

b. Compute the conjugate g^*. (1 p)

c. Can the primal problem be solved using the proximal gradient method? (1 p)

d. Can the dual problem be solved using the proximal gradient method? (You don't need to have solved **a.** or **b.** to answer this.) (1 p)

e. Assume a dual solution $\mu^\star$ is known. Provide *one* simple condition that recovers a primal solution. (You don't need to have solved **a.** or **b.** to answer this, you can phrase the recovery in terms of functions f or g, or conjugates f^* and g^*.) (1 p)

5. Consider the following constrained optimization problem

$$\underset{x}{\text{minimize}}\, f(x) + \iota_C(x),$$

where constraint qualification holds so that $x^\star$ solves the problem if and only if $0 \in \partial f(x^\star) + \partial \iota_C(x^\star)$. Assume you have solved the dual problem

$$\underset{\mu}{\text{minimize}}(f^*(\mu) + g^*(-\mu))$$

and found the optimal dual variable $\mu^\star = 0$. Show that all primal optimal $x^\star$ are also solutions to the unconstrained minimization problem

$$\text{minimize}\, f(x)$$

using primal-dual optimality conditions and Fermat's rule. (1 p)

6. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is β-smooth for some $\beta > 0$. Show that one step of the gradient method $x_{k+1} = x_k - \gamma \nabla f(x_k)$ gives the following function value decrease guarantee

$$f(x_{k+1}) - f(x_k) \leq -\gamma(1 - \tfrac{\beta\gamma}{2})\|\nabla f(x_k)\|_2^2.$$

Find the γ that gives the largest guaranteed function value decrease at any given iteration. (1 p)