Financial Risk
4-th quarter 2020/21
Lecture1: Financial risk, extreme value statistics

"As an alternative to the traditional 30-year mortgage, we also offer an interest-only mortgage, balloon mortgage, reverse mortgage, upside down mortgage, inside out mortgage, loop-de-loop mortgage, and the spinning double axel mortgage with a triple lutz."

Gudrun January 2005
326 MEuro loss
72 % due to forest losses
4 times larger than second largest

This course:

- Learn some fun things about financial risks
- Learn some basic risk management tools from Extreme Value Statistics
- See some basic quantitative Credit Risk models

But not

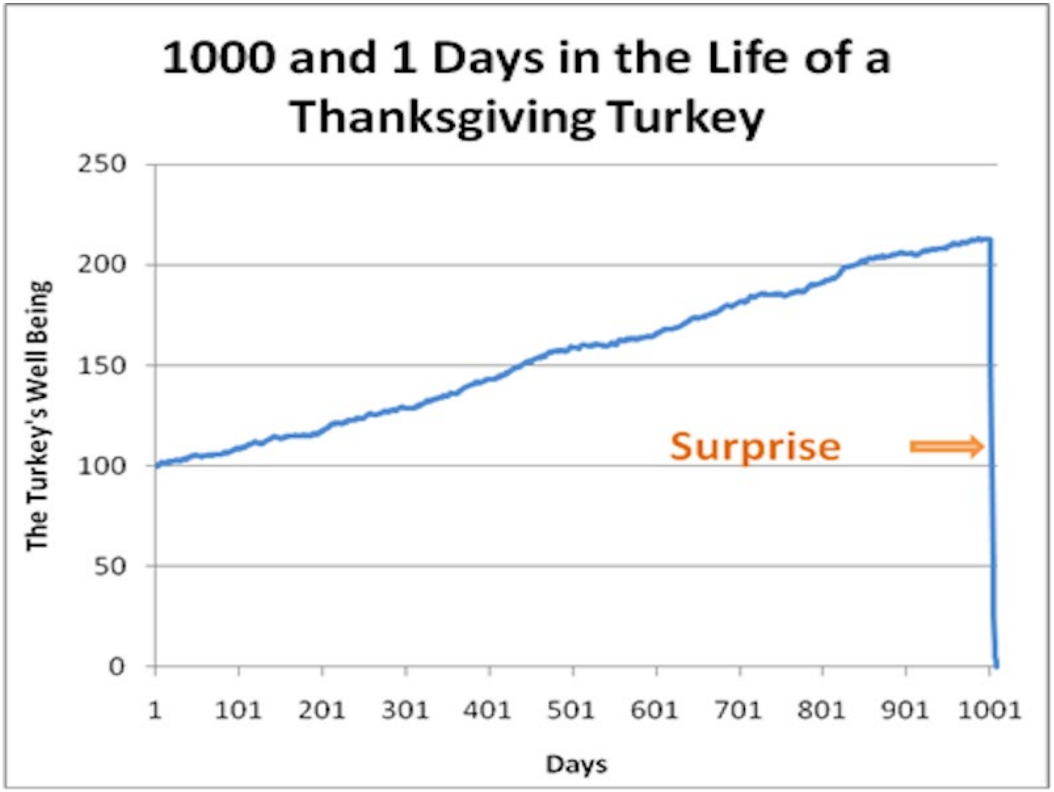
- A complete systematic account of financial risk management
- Financial time series modeling
- Black-Scholes option pricing methods
- Macroeconomics

And

There are risks which cannot be handled by mathematical models!!

Risk: event or action which prevents an institution from meeting its obligations or reaching its goals.

If one does not understand the real-world situation well enough, the best quantitative tools will not help. Taleb's Turkey example:



Real world statistics

Model checking and adjustment

- Independence
- Stationarity
- Distributional fit
- Period and threshold choice

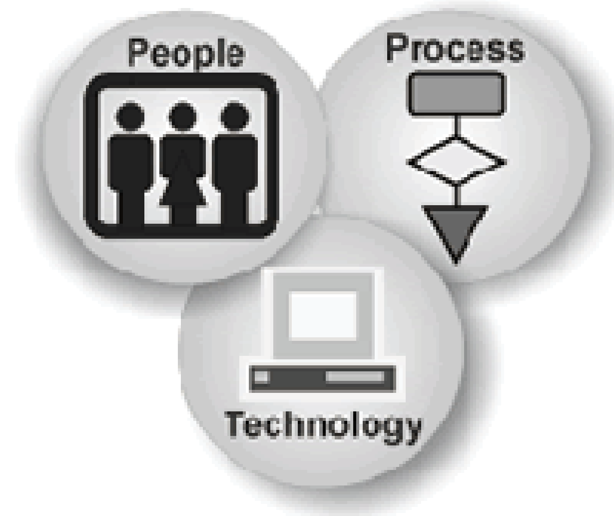
Statistical model ⇄ Data analysis

- Choice of distributions
- Estimation (often of quantiles)
- Confidence intervals
- "Prediction intervals"
- Hypothesis testing
- Regression and dependence modeling ***(not this course)***
- Prediction ***(not this course)***
- ***Understanding!!***

Refresh your basic statistics knowledge!

- Random variable
- Distribution functions
- Independent events, conditional probabilities
- Poisson process
- Expected value, variance, moments
- Correlation
- Point estimation
- Confidence intervals
- qq-plots, see http://data.library.virginia.edu/understanding-q-q-plots/ (*R* is a free software environment for statistical computing and graphics)

- Credit risk
- Market risk
- Operational risk
- Insurance risk
- Liquidity risk
- Reputational risk
- Legal risk
- and so on …



Operational risk

Market risk: risk that the value of a portfolio changes due to changes of market prices, exchange rates etc.

Credit risk: risk that the value of a portfolio changes because a debtor cannot meet his obligations.

Operational risk: risk caused by problems in internal processes, people, systems

Basel III: A global, voluntary regulatory framework on bank capital adequacy, stress testing, and market liquidity risk

Solvency 2: A Directive in European Union law that codifies and harmonises the EU insurance regulation. Primarily this concerns the amount of capital that EU insurance companies must hold to reduce the risk of insolvency.

Risk factors

L= "-P/L" = Loss – Profit =$F(X_1, \ldots X_d)$, $X_1, \ldots X_d$, risk factors, *e.g.* exchange rates, interest rates, index movements, stock prices, ….

Example: Linear portfolio, α_i # shares of stock *i*, stock price $S_{t,i}$

$$L = -\sum_{i=1}^{d} \alpha_i S_{t+1,i} + \sum_{i=1}^{d} \alpha_i S_{t,i} = -\sum_{i=1}^{d} \alpha_i S_{t,i} \left(\frac{S_{t+1,i} - S_{t,i}}{S_{t,i}} \right)$$

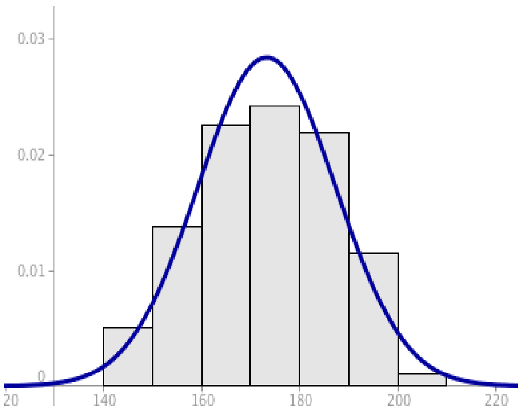
How big is the risk?

Quantitative risk management methods:

- Historical data or historical simulation
- Stress testing ("scenarios")
- Sensitivity measures ("the greeks")
- Full statistical modeling (often multivariate normal + linear portfolio)
- Semiparametric modeling of the "tails" of the *loss-profit* distribution (univariate EVS) ***(this course)***
- Semiparametric modeling of the tails of the multivariate distribution of the risk factors (multivariate Extreme Value Statistics, "Copulas") + computation of the *loss-profit* distribution analytically or via stochastic simulation

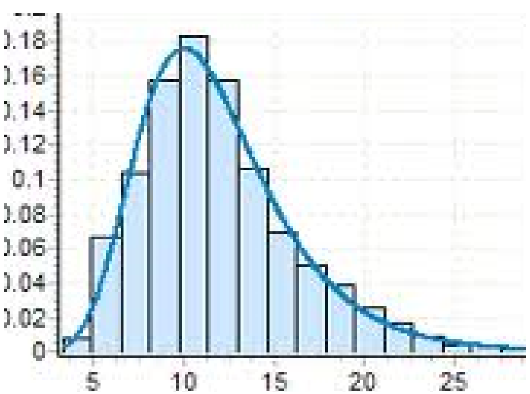
How big is the risk?

Matematics → shapes of possible risk distributions
Statistics → choice of specific risk distribution (with uncertainty)
Risk distribution → quantifies risk, perhaps via VaR or ES, perhaps multivariate



Normal distribution

Error in the measured distance = sum of many small measurement errors



Generalized Extreme Value distribution

Highest water level during year = maximum of daily water levels

Extreme value statistics (EVS) is the branch of statistics developed to handle extreme risks

The philosophy of EVS is simple: extreme events, perhaps extreme water levels or extreme financial losses, are often quite different from ordinary everyday behavior, and ordinary behavior then has little to say about extremes, so that only other extreme events give useful information about future extreme events.

Basic EVS:
--- Block Maxima: GEV distribution for maxima
--- Peaks over Thresholds: GP distribution for tails

Why?

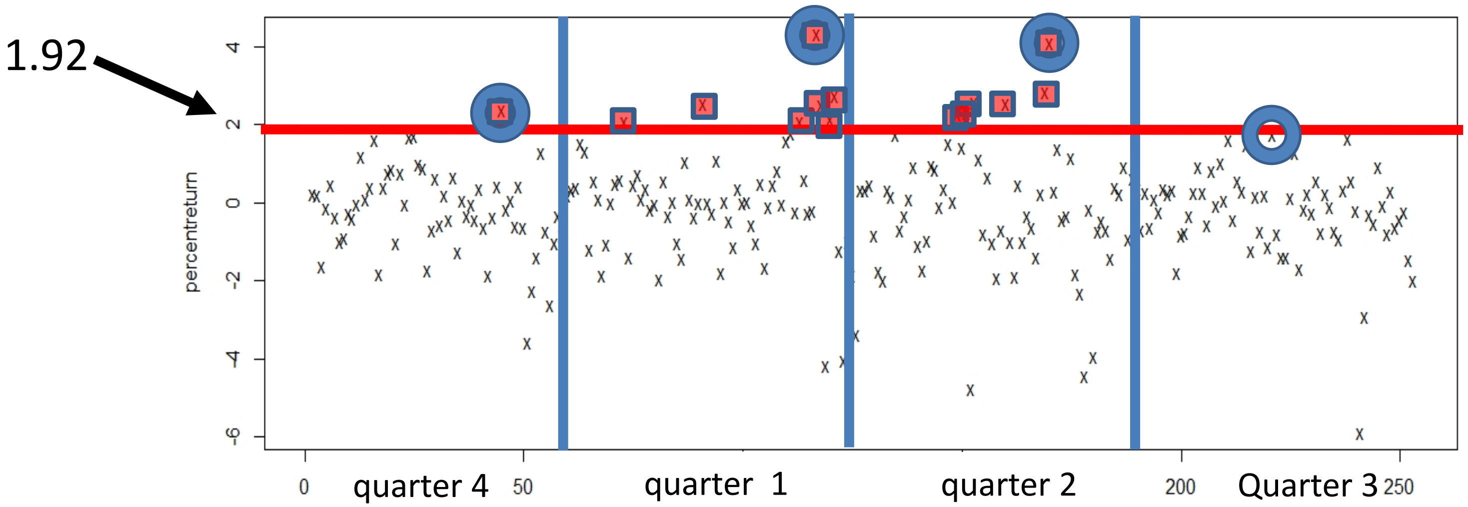
- **stability:** maxima of variables which are GEV distributed are also GEV; going to higher levels preserves the GP distribution of exceedances (cf. “standard statistics: sums of normally distributed variables have a normal distribution)

- **asymptotics:** maxima of many independent variables are often (approximately) GEV distributed; asymptotically tails are GP when maxima are EV (cf. “standard statistics: sums of many small “errors” are often (approximately) normally distributed the “central limit theorem”)

- **”transition”:** easy to go back and forth between GP and GEV

but don’t believe in models blindly

Apple losses (= - 100 $\times \frac{\text{price tomorrow} - \text{price today}}{\text{price today}}$) one year back



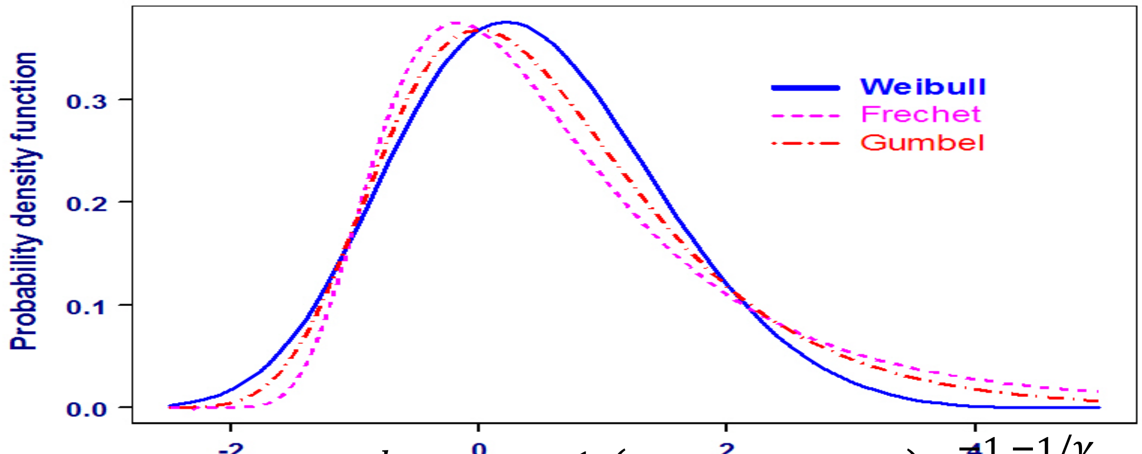
Maximum quarterly loss excess of the level $u = 1.92$

How large is the risk of a big quarterly loss? BM

How large is the risk of a big loss tomorrow? PoT

Generalized extreme value (GEV) distributions

$$G(x) = e^{-\left(1+\gamma\frac{x-\mu}{\sigma}\right)_+^{-1/\gamma}}$$



Probability density g(x) = $\frac{d}{dx}G(x) = \frac{1}{\sigma}\left(1+\frac{\gamma}{\sigma}(x-\mu)\right)_+^{-1-1/\gamma}, \quad \gamma \neq 0$

$\gamma > 0$ Frechet distribution, finite left endpoint $\mu + \sigma/\gamma$,
heavytailed, $\gamma \geq \frac{1}{2}$ variance doesn't exist

$\gamma = 0$ Gumbel distribution, $G(x) = \exp\{-e^{-(x-\mu)/\sigma}\}$,
unbounded, density $\frac{1}{\sigma}e^{-(x-\mu)/\sigma}\exp\{-e^{-(x-\mu)/\sigma}\}$,

$\gamma < 0$ Weibull distribution, finite right endpoint $\mu + \sigma/|\gamma|$

The block maxima method *(Coles p. 45-53)*

Obtain observations $x_1, \dots, x_n$ of block maxima (e.g. weekly or yearly maxima)

- Assume observations are i.i.d and have a GEV distribution
- Use $x, \dots, x_n$ to estimate the GEV parameters
- Use the fitted GEV to compute estimates and confidence intervals for, e.g., quantiles of yearly maximum distribution

- Many estimation methods: ML, moment estimators, PWM, bias-corrected, …
- **ML has the major advantage that it gives standardized ways for including trends in parameters and for testing of submodels – and for handling truncation and censoring**
- Profile likelihood or bootstrap confidence intervals often preferable

Some mathematics behind the Block Maxima Method:

$X_1, X_2, \ldots$ independent identically distributed (i.i.d.) random variables with distribution function (d.f.) F, so $F(x) = P(X_1 \leq x)$

$\mathrm{M_n} = \max\{X_1, \ldots, X_n\}$ maximum of the first *n* variables (*e.g. n* = 30 and *X* = daily losses)

$$P(M_n \leq x) = P(X_1 \leq x, \ldots X_n \leq x)$$
$$= P(X_1 \leq x) \times \cdots \times P(X_1 \leq x) = F(x)^n$$

Exercise: Show that the GEV distributions are *max-stable, i.e.* that the maximum of *n* i.i.d. GEV-distributed variables also have an GEV distribution, *i.e.* that if $G(x) = \exp\{-\left(1 + \gamma \frac{x-\mu}{\sigma}\right)^{-1/\gamma}\}$ then there are μ_n, σ_n such that

$$G(x)^n = \exp\{-\left(1 + \gamma \frac{x - \mu_n}{\sigma_n}\right)^{-\frac{1}{\gamma}}\}$$

and find μ_n, σ_n.

Theorem: *The distribution function G is max-stable if and only if it is an GEV distribution*

In the previous exercise it was shown that the EV distributions are max-stable, which proves half of this theorem. The other half consists of solving the functional equations

$$G(x)^n = G(\tfrac{x-\mu_n}{\sigma_n}), \ \text{for} \ n = 1, 2, \ldots$$

to find that the GEV distributions are the only solutions.

Theorem: *If there are constants* $b_n > 0, a_n$ *such that*

$$P\left(\tfrac{M_n - a_n}{b_n} \leq x\right) \rightarrow G(x), \ \text{as} \ n \rightarrow \infty \ \text{for all} \ x$$

then G(x) is an GEV distribution. ***(cf. the central limit theorem)***

This is proved by showing that it follows that *G(x)* must be max-stable

Exercise: Suppose X_1, X_2, … are i.i.d. and Pareto distributed random variables with distribution function (d.f.)

$$F(x) = 1 - \left(\frac{K}{x}\right)^{\alpha}, \qquad x \geq K, \qquad K, \alpha > 0,$$

Let $a_n = Kn^{1/\alpha}, b_n = Kn^{1/\alpha}$. Then

$$P\left(\frac{M_n - a_n}{b_n} \leq x\right) = P(M_n \leq b_n x + a_n)$$

$$= F(b_n x + a_n)^n = \left\{1 - \left(\frac{K}{b_n x + a_n}\right)^{\alpha}\right\}^n$$

$$= \left\{1 - \frac{1}{n}\left(\frac{1}{x+1}\right)^{\alpha}\right\}^n \to e^{-(1+x)^{-\alpha}} \;\; as\; n \to \infty,$$

since $\left(1 + \frac{a}{n}\right)^n \to e^a \;\; as\; n \to \infty$.

(A perhaps unnecessary explanation) **What does**

$$P\left(\frac{M_n - a_n}{b_n} \leq x\right) \to G(x), \quad \text{as} \quad n \to \infty \quad \text{for all} \quad x,$$

mean in practice? That $P\left(\frac{M_n - a_n}{b_n} \leq x\right) \approx G(x)$, for large n, or,

with $y = b_n x + a_n$ and $G(x) = \exp\{-(1 + \gamma \frac{x - \mu'}{\sigma'}))^{-1/\gamma}\}$, that

$$\begin{aligned} P\left(M_n \leq y\right) \quad &\approx \quad G\left(\frac{y - a_n}{b_n}\right) = \exp\{-(1 + \gamma \frac{y - (a_n + b_n \mu')}{b_n \sigma'})^{-1/\gamma}\} \\ &= \quad \exp\{-(1 + \gamma \frac{y - \mu}{\sigma})^{-1/\gamma}\}, \text{ for } \mu = a_n + b_n \mu', \ \sigma = b_n \sigma. \end{aligned}$$

Since all the parameters are unknown anyway, we are left with the problem of estimating μ, σ, γ from data, *i.e.* to use the Block Maxima method.