Fourieranalys MVE030 och Fourier Metoder MVE290 20.mars.2020
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA.
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1. Solve:
$$\begin{cases} u_{tt} - u_{xx} = f(t)g(x) & 0 < t, -1 < x < 1 \\ u(x,0) = \varphi(x) = u_t(x,0) \\ u(-1,t) = 0, \quad u_x(1,t) = 5 \end{cases}$$

   So, apparently the exam is going to be a little unusual, but you can continue studying exactly as you're all doing, and this should prepare you well!

   This problem, yikes, where to begin. Well, we must deal with that inhomogeneous boundary condition first. We do this by finding a steady state solution to solve the homogeneous PDE, this pesky 5 condition, and keep the other nice homogeneous boundary condition. Thus we seek F to satisfy
$$-F''(x) = 0, \quad F(-1) = 0, \quad F'(1) = 5.$$
   The solution is
$$F(x) = 5x + 5.$$
   Next we solve a new problem which we like better:
$$\heartsuit\heartsuit : \begin{cases} w_{tt} - w_{xx} = 0 \\ w(x,0) = \varphi(x) - F(x) \\ w_t(x,0) = \varphi(x) \\ w(-1,t) = 0 = w_x(1,t). \end{cases}$$
   To do this we separate variables. Write $w = TX$. Put into the PDE. This gives
$$T''X - X''T = 0 \iff \frac{T''}{T} = \frac{X''}{X} \implies \text{both sides are constant.}$$

We consider all the cases for this constant, which we name λ. In doing this, we solve for X first because it has homogeneous boundary conditions:

$$X(-1) = 0, \quad X'(1) = 0.$$

So, the equation for X is

$$X'' = \lambda X.$$

If $\lambda = 0$ then X is a linear function. The condition $X'(1) = 0$ shows that X is constant. The condition $X(-1) = 0$ shows that $X \equiv 0$. This is not interesting. If $\lambda > 0$ then our function is a linear combination

$$Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}.$$

The boundary conditions require

$$Ae^{-\sqrt{\lambda}} + Be^{\sqrt{\lambda}} = 0, \quad \sqrt{\lambda}\left(Ae^{\sqrt{\lambda}} - Be^{-\sqrt{\lambda}}\right) = 0.$$

Since $\lambda > 0$, we can divide by it in the second equation, obtaining the equation

$$Ae^{\sqrt{\lambda}} - Be^{-\sqrt{\lambda}} = 0.$$

Let us multiply our two equations by $e^{\sqrt{\lambda}}$ obtaining the two equations

$$A + Be^{2\sqrt{\lambda}} = 0 \implies A = -Be^{2\sqrt{\lambda}},$$

and

$$Ae^{2\sqrt{\lambda}} - B = 0 \implies A = Be^{-2\sqrt{\lambda}}.$$

If $A > 0$, then $A = -Be^{2\sqrt{\lambda}}$ implies $B < 0$, but $A = Be^{-2\sqrt{\lambda}}$ implies $B > 0$. Contradictions abound! If $A < 0$, then $A = -Be^{2\sqrt{\lambda}}$ implies $B > 0$, but $A = Be^{-2\sqrt{\lambda}}$ implies $B < 0$. Contradictions abound! Thus the only solution we find in this case is (just like I told y'all in class) is the zero solution.

So, we look for solutions with $\lambda < 0$. These can be expressed with sine and cosine and will be a linear combination as such. Write

$$A\cos(\sqrt{|\lambda|}x) + B\sin(\sqrt{|\lambda|}x).$$

We investigate the boundary conditions, for this the evenness and oddness of cosine and sine, respectively are useful properties. The first boundary condition (at $x = -1$) requires

$$A\cos\sqrt{|\lambda|} - B\sin\sqrt{|\lambda|} = 0,$$

whereas the second boundary condition (at $x = 1$) requires

$$-\sqrt{|\lambda|}A \sin\sqrt{|\lambda|} + \sqrt{|\lambda|}B \cos\sqrt{|\lambda|} = 0.$$

Since $\sqrt{|\lambda|} \neq 0$ we may divide by it in this equation obtaining

$$-A \sin\sqrt{|\lambda|} + B \cos\sqrt{|\lambda|} = 0.$$

So, now we obtain by first adding the two equations

$$(A + B)\cos\sqrt{|\lambda|} - (A + B)\sin\sqrt{|\lambda|} = 0.$$

Subtracting the equations we obtain:

$$(A - B)\cos\sqrt{|\lambda|} + (A - B)\sin\sqrt{|\lambda|} = 0.$$

These two equations are satisfied in two different cases. First case $A+B = 0$ so that $B = -A$, in which case the first equation is satisfied and the second equation requires

$$\cos\sqrt{|\lambda|}+\sin\sqrt{|\lambda|} = 0 \implies \sqrt{|\lambda|} = \sqrt{|\lambda|}_n = \nu_n = \frac{4n+3}{4}\pi, \quad n \in \mathbb{N}_0.$$

Second case we have $A - B = 0$ so that $B = A$, in which case the second equation is satisfied and the first equation requires

$$\cos\sqrt{|\lambda|}-\sin\sqrt{|\lambda|} = 0 \implies \sqrt{|\lambda|} = \sqrt{|\lambda|}_n = \nu_n = \frac{4n+1}{4}\pi, \quad n \in \mathbb{N}_0.$$

Without loss of generality we may take the constants to be ± 1, since the constants shall come from the initial conditions, thus

$$X_n(x) = \begin{cases} \cos(\nu_n x) - \sin(\nu_n x) & \nu_n = \frac{4n+3}{4}\pi, \quad \lambda_n = -\frac{(4n+3)^2\pi^2}{4^2} \\ \cos(\nu_n x) + \sin(\nu_n x) & \nu_n = \frac{4n+1}{4}\pi \quad \lambda_n = -\frac{(4n+1)^2\pi^2}{4^2}. \end{cases}$$

Now that we have these values, we have the equation for

$$T_n'' = \lambda_n T_n \implies T_n(t) = a_n \cos(\nu_n t) + b_n \sin(\nu_n t).$$

We smash the solutions together into a supersolution (by the superposition principle) to obtain

$$w(x,t) = \sum_{n \geq 0} X_n(x) T_n(t).$$

To determine the a_n and b_n, we use the initial conditions. Setting $t = 0$ we have

$$w(x,0) = \sum_{n\geq 0} a_n X_n(x) \implies a_n = \frac{\int_{-1}^{1} (\varphi(x) - F(x))\overline{X_n(x)}dx}{\int_{-1}^{1} |X_n(x)|^2 dx}.$$

This is because we want $w(x,0) = \varphi(x) - F(x)$. Next we compute the derivative at $t = 0$,

$$w_t(x,0) = \sum_{n\geq 0} b_n \nu_n X_n(x) \implies b_n = \frac{\int_{-1}^{1} \varphi(x)\overline{X_n(x)}dx}{\nu_n \int_{-1}^{1} |X_n(x)|^2 dx}.$$

This is because we want $w_t(x,0) = \varphi(x)$. So now we have solved for w.

The last piece of the puzzle is to solve:

$$\begin{cases} v_{tt} - v_{xx} = f(t)g(x) \\ v(x,0) = 0 = v_t(x,0) \\ v(-1,t) = 0 = v(1,t) \end{cases}$$

To do this we make the ansatz that we can find a solution of the form

$$v(x,t) = \sum_{n\geq 0} c_n(t) X_n(x),$$

for some functions $c_n(t)$. Using X_n in this way we guarantee that the beautiful (self-adjoint, homogeneous, oh yay) boundary conditions are satisfied. We put this into the PDE:

$$\sum_{n\geq 0} c_n''(t)X_n(x) - c_n(t)X_n''(x) = \sum_{n\geq 0} X_n(x)\left(c_n''(t) - \lambda_n c_n(t)\right).$$

Above, we used the fact that X_n satisfies the (regular!) SLP:

$$X_n''(x) = \lambda_n X_n(x).$$

On the other side of the PDE we Fourier expand in the x variable,

$$f(t)g(x) = \sum_{n\geq 0} f(t)\hat{g}_n X_n(x), \quad \hat{g}_n = \frac{\int_{-1}^{1} g(x)\overline{X_n(x)}dx}{\int_{-1}^{1} |X_n(x)|^2 dx}.$$

So, we equate these two series:

$$\sum_{n\geq 0} X_n(x)\left(c_n''(t) - \lambda_n c_n(t)\right) = \sum_{n\geq 0} f(t)\hat{g}_n X_n(x).$$

Then we equation the coefficients of $X_n(x)$, requiring therefore that

$$c_n''(t) - \lambda_n c_n(t) = f(t)\hat{g}_n, \quad c_n(0) = 0 = c_n'(0).$$

The last two conditions are because we want $v(x,0) = 0 = v_t(x,0)$. Now, since we have no explicit expressions for $f(t)$ or $g(x)$, we cannot explicitly solve this ODE. However, general ODE theory says that a solution exists (as long as $f(t)$ is some decent function), and that it is unique. Thus $c_n(t)$ should be this solution to this ODE, for each n. Having specified F, w, and v at this point, our solution to the original problem is

$$u(x,t) = F(x) + w(x,t) + v(x,t).$$

2. Use the Fourier series expansion of $\cos(\alpha t)$ on $(-\pi, \pi)$ to compute for $|\alpha| < 1$

$$\prod_{n\geq 1} \frac{n^2 - \alpha^2}{n^2}.$$

Hint: The Fourier series is

$$\frac{\sin(\alpha\pi)}{\pi}\left(\frac{1}{\alpha} + 2\alpha \sum_{n\geq 1} \frac{(-1)^{n+1}\cos(nt)}{n^2 - \alpha^2}\right).$$

So, let's start by making the oscillation go away, which can be achieved by choosing $t = \pi$. It is clear we will have to set t equal to something because the statement about the product has no tea[1] So, if we set $t = \pi$ then the Fourier series converges to the average of the left and right limits of $\cos(\alpha t)$, giving

$$\frac{\cos(-\pi\alpha) + \cos(\pi\alpha)}{2} = \frac{\sin(\alpha\pi)}{\pi}\left(\frac{1}{\alpha} + 2\alpha \sum_{n\geq 1} \frac{(-1)^{n+1}\cos(n\pi)}{n^2 - \alpha^2}\right).$$

[1] It also has no shade. RuPaul reference.

We use the fact that cos is even and $\cos(n\pi) = (-1)^n$ to simplify this to:

$$\cos(\pi\alpha) = \frac{\sin(\alpha\pi)}{\pi}\left(\frac{1}{\alpha} - 2\alpha\sum_{n\geq 1}\frac{1}{n^2-\alpha^2}\right).$$

Let us tidy this up,

$$\frac{\pi\cos(\pi\alpha)}{\sin(\alpha\pi)} - \frac{1}{\alpha} = \sum_{n\geq 1}\frac{-2\alpha}{n^2-\alpha^2}.$$

Now, if we just look at this, we see functions which are derivatives of logs (with respect to α). See:

$$\frac{d}{d\alpha}(\log(\sin(\alpha\pi))) = \frac{\pi\cos(\pi\alpha)}{\sin(\alpha\pi)}, \quad \frac{d}{d\alpha}(\log(1/\alpha)) = -\frac{1}{\alpha}.$$

Similarly

$$\frac{d}{d\alpha}\log(n^2-\alpha^2) = \frac{-2\alpha}{n^2-\alpha^2}.$$

So, we have basically the equation:

$$\frac{d}{d\alpha}(\log(\sin(\alpha\pi)) + \log(1/\alpha)) = \sum_{n\geq 1}\frac{d}{d\alpha}\log(n^2-\alpha^2).$$

The fact that we now have logs is good because logs can turn sums into products, since $\log a + \log b = \log(ab)$. So we have a chance of getting an infinite product if we can say that the two sides are equal once we remove the derivatives. There are two problems. First, what is the constant of integration? If we remove the derivatives, we know that the two sides are equal up to a constant factor. What should that be? Second, uh, if we remove the derivatives under the sum, we get something that does *not* converge. This is not good. So, let's think about, since $|\alpha| < 1$, what happens if $\alpha \to 0$,

$$\lim_{\alpha\to 0}\log(\sin(\alpha\pi)) + \log(1/\alpha) = \lim_{\alpha\to 0}\log\left(\frac{\sin(\alpha\pi)}{\alpha}\right) = \log(\pi).$$

On the other side,

$$\lim_{\alpha\to 0}\sum_{n\geq 1}\log(n^2-\alpha^2) = \sum_{n\geq 1}\log(n^2) \to \infty.$$

So, we would like to make the right side finite, but we can do this because we can add anything that doesn't depend on α to $\log(n^2+\alpha^2)$ and the derivative remains the same. So, let's be smart about this and add $-\log(n^2)$. Then the right becomes

$$\lim_{\alpha\to 0}\sum_{n\geq 1}\log(n^2-\alpha^2)-\log(n^2)=0.$$

Moreover, our sum converges for $|\alpha|<1$

$$\sum_{n\geq 1}\log(n^2-\alpha^2)-\log(n^2)=\sum_{n\geq 1}\log\left(\frac{n^2-\alpha^2}{n^2}\right).$$

So, letting $\alpha\to 0$ on the right we now get zero, but on the left we are getting $\log\pi$, so this indicates that the constant of integration on the left should be $-\log\pi$. Thus, we have the nice and rigorous equality

$$\log(\sin(\alpha\pi))+\log(1/\alpha)-\log\pi=\sum_{n\geq 1}\log\left(\frac{n^2-\alpha^2}{n^2}\right).$$

On the left we can put everything together:

$$\log\left(\frac{\sin(\alpha\pi)}{\alpha\pi}\right)=\sum_{n\geq 1}\log\left(\frac{n^2-\alpha^2}{n^2}\right).$$

Now we can exponentiate both sides, obtaining

$$\frac{\sin(\alpha\pi)}{\alpha\pi}=\prod_{n\geq 1}\frac{n^2-\alpha^2}{n^2}.$$

Nifty!

3. Compute

$$\int_0^\infty\frac{\sin(\sqrt{t})}{e^{2t}}dt.$$

The very first thing I would like to do with this is make that $\sqrt{t}$ go away! Square roots are intrinsically complicated things. So, let's do a variable substitution letting

$$u=\sqrt{t}\implies du=\frac{1}{2\sqrt{t}}dt=\frac{1}{2u}dt.$$

The limits of integration don't change, so we are computing

$$\int_0^\infty \sin(u)(2u)e^{-2u^2}du.$$

I would like to use some Fourier methods, for this I'd like the integral to be over $\mathbb{R}$. This can be achieved since the integrand is an even function so

$$\heartsuit := \int_0^\infty \sin(u)(2u)e^{-2u^2}du = \int_0^\infty u\sin(u)e^{-2u^2}du.$$

Now, we can simplify this because

$$\frac{d}{du} - \frac{1}{4}e^{-2u^2} = ue^{-2u^2}.$$

So, integrating by parts we have

$$\heartsuit = -\frac{1}{4}e^{-2u^2}\sin(u)\Big|_{-\infty}^{\infty} + \int_{\mathbb{R}} \frac{1}{4}e^{-2u^2}\cos(u)du.$$

The first term vanishes due to the presence of a GODZILLA term, e^{-2u^2}. As for the second term, we remember that

$$\cos(u) = \frac{1}{2}\left(e^{iu} + e^{-iu}\right),$$

so

$$\heartsuit = \frac{1}{8}\int_{\mathbb{R}} e^{-2u^2}e^{iu}du + \frac{1}{8}\int_{\mathbb{R}} e^{-2u^2}e^{-iu}du.$$

The first integral is the Fourier transform of e^{-2u^2} at -1, and the second term is the Fourier transform at 1. The Fourier transform is provided to us in a table. The Fourier transform of $e^{-ax^2/2}$ is $\sqrt{2\pi/a}e^{-\xi^2/2a}$. For our case, $a = 4$. Thus

$$\heartsuit = \frac{1}{8}\sqrt{2\pi/4}e^{-1/8} + \frac{1}{8}\sqrt{2\pi/4}e^{-1/8}.$$

We could simplify this if we wanted to do it.

Alternative solution: you are cleverer than I am and you swiftly observe that

$$\heartsuit = \mathcal{L}(\sin(\sqrt{t}))(-2).$$

That is the Laplace transform of $\sin(\sqrt{t})$ evaluated at $z = -2$. Perhaps you have this sitting in the Beta book and can read it off a table there. Hopefully the formula in the table is right in that case!

4. Solve:

$$\begin{cases} u(0,t) = f(t) & t > 0 \\ u_t(x,t) - u_{xx}(x,t) = 0 & t, x > 0 \\ u(x,0) = 0 & x > 0 \end{cases}$$

This problem just begs us to use the Laplace transform. It's a homogeneous heat equation with a fire at $x = 0$. So, let's not disappoint and apply the Laplace transform in the t variable:

$$z\widetilde{u}(x,z) - \widetilde{u}_{xx}(x,z) = 0 \implies \widetilde{u}(x,z) = a(z)e^{x\sqrt{z}} + b(z)e^{-x\sqrt{z}}.$$

Here we have used the fact that

$$\widetilde{u_t}(x,z) = z\widetilde{u}(x,z) - u(x,0) = z\widetilde{u}(x,z).$$

Now, to figure out the $a(z)$ and $b(z)$ we use the boundary condition. Laplace transforming the boundary condition we get

$$\widetilde{u}(0,z) = \widetilde{f}(z) \implies a(z) + b(z) = \widetilde{f}(z).$$

Super. We would like to go backwards to find the solution before it got Laplace transformed. Well, if I look in my tables I can certainly find a function whose Laplace transform is $e^{-x\sqrt{z}}$. Is there a function whose Laplace transform is $e^{x\sqrt{z}}$? No, there is not. Furthermore, for any function that is Laplace transformable, we have that its Laplace transform tends to ZERO as $\Re(z) \to \infty$. Since $x > 0$ in this problem, what happens to $e^{x\sqrt{z}}$ if $\Re(z) \to \infty$? It tends to infinity. With nearly Godzilla like speed. It certainly does NOT tend to zero. So there is no Laplace-transformable function whose Laplace transform is $e^{x\sqrt{z}}$. Hence, we would like to find a solution that does not require this part. Can we do it? Well, yes we can. By our tables and properties of Laplace transform, the convolution of

$$f(t)\Theta(t) \text{ with } \frac{\Theta(t)t^{-3/2}x}{2\sqrt{\pi}}e^{-x^2/(4t)} \text{ in the } t \text{ variable}$$

has Laplace transform equal to

$$\widetilde{f}(z)e^{-x\sqrt{z}}.$$

So, we have found that

$$u(x,t) = \int_{\mathbb{R}} f(t-s)\Theta(t-s)\frac{\Theta(s)s^{-3/2}x}{2\sqrt{\pi}}e^{-x^2/(4s)}ds.$$

In case we have forgotten, Θ is the heavyside function, which vanishes whenever its argument is negative, and which is 1 whenever its argument is positive. Thus

$$u(x,t) = \int_0^t f(t-s)s^{-3/2}\frac{x}{2\sqrt{\pi}}e^{-x^2/(4s)}ds.$$

5. This could be a PDE in a half space with a nice boundary condition (extend evenly or oddly!). Another reasonable candidate is computing an integral with help of Fourier transform (like those EÖ number 7-12). Or an integral equation where you use convolution and Fourier transform (like EÖ 13, 14). Or an SLP (EÖ 23, 24). Or a PDE in a box (EÖ 25), or in a disk, or a wedge. Or a best approximation.

6. This could be a best approximation, or an SLP, or maybe something involving Bessel functions. Or possibly something to test conceptual understanding without actually needing to do much calculating.

7. Prove a theory item!

8. Prove a theory item!

Fourier transformer (Fourier transforms) där $a > 0$ och $c \in \mathbb{R}$.

$f(x)$	$\hat{f}(\xi)$
$f(x-c)$	$e^{-ic\xi}\hat{f}(\xi)$
$e^{ixc}f(x)$	$\hat{f}(\xi-c)$
$f(ax)$	$a^{-1}\hat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\hat{f}(\xi)$
$xf(x)$	$i(\hat{f})'(\xi)$
$(f*g)(x)$	$\hat{f}(\xi)\hat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\hat{f}*\hat{g})(\xi)$
$e^{-ax^2/2}$	$\sqrt{2\pi/a}e^{-\xi^2/(2a)}$
$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\|\xi\|}$
$e^{-a\|x\|}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x) = \begin{cases} 1 & \|x\| < a \\ 0 & \|x\| > a \end{cases}$	$2\xi^{-1}\sin(a\xi)$
$x^{-1}\sin(ax)$	$\pi\chi_a(\xi) = \begin{cases} \pi & \|\xi\| < a \\ 0 & \|\xi\| > a \end{cases}$

Laplace transformer (Laplace transforms) där $a > 0$, $c \in \mathbb{C}$, och

$$H(t) := \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$H(t)f(t)$	$\widetilde{f}(z)$
$H(t-a)f(t-a)$	$e^{-az}\widetilde{f}(z)$
$H(t)e^{ct}f(t)$	$\widetilde{f}(z-c)$
$H(t)f(at)$	$a^{-1}\widetilde{f}(a^{-1}z)$
$H(t)f'(t)$	$z\widetilde{f}(z) - f(0)$
$H(t)\int_0^t f(s)ds$	$z^{-1}\widetilde{f}(z)$
$H(t)(f * g)(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
$H(t)t^{-1/2}e^{-a^2/(4t)}$	$\sqrt{\pi/z}e^{-a\sqrt{z}}$
$H(t)t^{-3/2}e^{-a^2/(4t)}$	$2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$
$H(t)J_0(\sqrt{t})$	$z^{-1}e^{-1/(4z)}$
$H(t)\sin(ct)$	$c/(z^2+c^2)$
$H(t)\cos(ct)$	$z/(z^2+c^2)$
$H(t)e^{-a^2t^2}$	$(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\operatorname{erfc}(z/(2a))$
$H(t)\sin(\sqrt{at})$	$\sqrt{\pi a/(4z^3)}e^{-a/(4z)}$

Lycka till! May the force be with you! ♡ Julie Rowlett, Carl-Joar Karlsson, Joao Pedro Paulos, Erik Jansson, Kolya Pochekai