Slides for additive vs. Linear functions

Old exercise which we presented.

If $E \subset R$ has positive measure, then $E - E$ contains an open interval around 0.
We will use this to prove an interestiong theorem.

Linear and Additive functions

As we know, $f : R \rightarrow R$ is linear if

Linear and Additive functions

As we know, $f : R \to R$ is linear if

1. $f(x+y) = f(x) + f(y)$ for all $x, y \in R$

Linear and Additive functions

As we know, $f : R \to R$ is linear if

$$1.\ f(x+y) = f(x) + f(y) \text{ for all } x, y \in R$$

and

$$2.\ f(cx) = cf(x) \text{ for all } x, c \in R.$$

Linear and Additive functions

As we know, $f : R \to R$ is linear if

1. $f(x+y) = f(x) + f(y)$ for all $x, y \in R$

and

2. $f(cx) = cf(x)$ for all $x, c \in R$.

In fact, in 1-d, it is easy to check that (2) implies (1).

Linear and Additive functions

As we know, $f : R \to R$ is linear if

1. $f(x+y) = f(x) + f(y)$ for all $x, y \in R$

and

2. $f(cx) = cf(x)$ for all $x, c \in R$.

In fact, in 1-d, it is easy to check that (2) implies (1).
The question here is whether (1) implies (2).

Linear and Additive functions

As we know, $f : R \to R$ is linear if

1. $f(x+y) = f(x) + f(y)$ for all $x, y \in R$

and

2. $f(cx) = cf(x)$ for all $x, c \in R$.

In fact, in 1-d, it is easy to check that (2) implies (1).
The question here is whether (1) implies (2).

We call f additive if it satisfies (1).

Linear and Additive functions

As we know, $f : R \to R$ is linear if

1. $f(x + y) = f(x) + f(y)$ for all $x, y \in R$

and

2. $f(cx) = cf(x)$ for all $x, c \in R$.

In fact, in 1-d, it is easy to check that (2) implies (1).
The question here is whether (1) implies (2).

We call f additive if it satisfies (1). This is the same as saying that f is a group homomorphism from R to R.

Linear and Additive functions

As we know, $f : R \to R$ is linear if

1. $f(x + y) = f(x) + f(y)$ for all $x, y \in R$

and

2. $f(cx) = cf(x)$ for all $x, c \in R$.

In fact, in 1-d, it is easy to check that (2) implies (1).
The question here is whether (1) implies (2).

We call f additive if it satisfies (1). This is the same as saying that f is a group homomorphism from R to R.

Does (1) imply (2)?

Linear and Additive functions

As we know, $f : R \to R$ is linear if

1. $f(x+y) = f(x) + f(y)$ for all $x, y \in R$

and

2. $f(cx) = cf(x)$ for all $x, c \in R$.

In fact, in 1-d, it is easy to check that (2) implies (1).
The question here is whether (1) implies (2).

We call f additive if it satisfies (1). This is the same as saying that f is a group homomorphism from R to R.

Does (1) imply (2)? Answer: No.

Linear and Additive functions

As we know, $f : R \to R$ is linear if

1. $f(x + y) = f(x) + f(y)$ for all $x, y \in R$

and

2. $f(cx) = cf(x)$ for all $x, c \in R$.

In fact, in 1-d, it is easy to check that (2) implies (1).
The question here is whether (1) implies (2).

We call f additive if it satisfies (1). This is the same as saying that f is a group homomorphism from R to R.

Does (1) imply (2)? Answer: No. But yes under the weak assumption of f being measurable.

Linear and Additive functions

Proposition: There exists $f : R \to R$ which is additive but not linear.

Linear and Additive functions

Proposition: There exists $f : R \to R$ which is additive but not linear.

Proof outline:

Linear and Additive functions

Proposition: There exists $f : R \to R$ which is additive but not linear.

Proof outline:
View R as a vector space over Q (which is then infinite dimensional).

Linear and Additive functions

Proposition: There exists $f : R \to R$ which is additive but not linear.

Proof outline:
View R as a vector space over Q (which is then infinite dimensional). One checks that the set $\{1, \sqrt{2}\}$ is linearly independent.

Linear and Additive functions

Proposition: There exists $f : R \to R$ which is additive but not linear.

Proof outline:
View R as a vector space over Q (which is then infinite dimensional). One checks that the set $\{1, \sqrt{2}\}$ is linearly independent. (This follows easily from the fact that $\sqrt{2}$ is irrational.)

Linear and Additive functions

Proposition: There exists $f : R \to R$ which is additive but not linear.

Proof outline:
View R as a vector space over Q (which is then infinite dimensional). One checks that the set $\{1, \sqrt{2}\}$ is linearly independent. (This follows easily from the fact that $\sqrt{2}$ is irrational.) Extend this set to a basis $\mathcal{B}$ for R (as a Q-vector space).

Linear and Additive functions

Proposition: There exists $f : R \to R$ which is additive but not linear.

Proof outline:
View R as a vector space over Q (which is then infinite dimensional). One checks that the set $\{1, \sqrt{2}\}$ is linearly independent. (This follows easily from the fact that $\sqrt{2}$ is irrational.) Extend this set to a basis $\mathcal{B}$ for R (as a Q-vector space). (Warning: Axiom of Choice.)

Linear and Additive functions

Proposition: There exists $f : R \to R$ which is additive but not linear.

Proof outline:
View R as a vector space over Q (which is then infinite dimensional). One checks that the set $\{1, \sqrt{2}\}$ is linearly independent. (This follows easily from the fact that $\sqrt{2}$ is irrational.) Extend this set to a basis $\mathcal{B}$ for R (as a Q-vector space). (Warning: Axiom of Choice.)

Let f be a function from $\mathcal{B}$ to R which takes 1 to 1, $\sqrt{2}$ to 3 and is arbitrary on the other elements of $\mathcal{B}$.

Linear and Additive functions

Proposition: There exists $f : R \to R$ which is additive but not linear.

Proof outline:
View R as a vector space over Q (which is then infinite dimensional). One checks that the set $\{1, \sqrt{2}\}$ is linearly independent. (This follows easily from the fact that $\sqrt{2}$ is irrational.) Extend this set to a basis $\mathcal{B}$ for R (as a Q-vector space). (Warning: Axiom of Choice.)

Let f be a function from $\mathcal{B}$ to R which takes 1 to 1, $\sqrt{2}$ to 3 and is arbitrary on the other elements of $\mathcal{B}$. By linear algebra, f can be extended to a Q-linear transformation from R to R,

Linear and Additive functions

Proposition: There exists $f : R \to R$ which is additive but not linear.

Proof outline:
View R as a vector space over Q (which is then infinite dimensional). One checks that the set $\{1, \sqrt{2}\}$ is linearly independent. (This follows easily from the fact that $\sqrt{2}$ is irrational.) Extend this set to a basis $\mathcal{B}$ for R (as a Q-vector space). (Warning: Axiom of Choice.)

Let f be a function from $\mathcal{B}$ to R which takes 1 to 1, $\sqrt{2}$ to 3 and is arbitrary on the other elements of $\mathcal{B}$. By linear algebra, f can be extended to a Q-linear transformation from R to R, meaning (1) holds and (2) holds for $c \in Q$.

Linear and Additive functions

Proposition: There exists $f : R \to R$ which is additive but not linear.

Proof outline:
View R as a vector space over Q (which is then infinite dimensional). One checks that the set $\{1, \sqrt{2}\}$ is linearly independent. (This follows easily from the fact that $\sqrt{2}$ is irrational.) Extend this set to a basis $\mathcal{B}$ for R (as a Q-vector space). (Warning: Axiom of Choice.)

Let f be a function from $\mathcal{B}$ to R which takes 1 to 1, $\sqrt{2}$ to 3 and is arbitrary on the other elements of $\mathcal{B}$. By linear algebra, f can be extended to a Q-linear transformation from R to R, meaning (1) holds and (2) holds for $c \in Q$. In particular, f is additive.

Linear and Additive functions

Proposition: There exists $f : R \to R$ which is additive but not linear.

Proof outline:
View R as a vector space over Q (which is then infinite dimensional). One checks that the set $\{1, \sqrt{2}\}$ is linearly independent. (This follows easily from the fact that $\sqrt{2}$ is irrational.) Extend this set to a basis $\mathcal{B}$ for R (as a Q-vector space). (Warning: Axiom of Choice.)

Let f be a function from $\mathcal{B}$ to R which takes 1 to 1, $\sqrt{2}$ to 3 and is arbitrary on the other elements of $\mathcal{B}$. By linear algebra, f can be extended to a Q-linear transformation from R to R, meaning (1) holds and (2) holds for $c \in Q$. In particular, f is additive. But f cannot be R-linear, since any such map $x \to ax$ which takes 1 to 1 is the identity.
QED

Linear and Additive functions

Theorem

If $f : R \to R$ *is additive and measurable, then* f *is linear.*

Linear and Additive functions

Theorem

If $f : R \to R$ is additive and measurable, then f is linear.

Before we prove this, we begin with a warm up which is weaker.

Linear and Additive functions

Theorem

If $f : R \to R$ is additive and measurable, then f is linear.

Before we prove this, we begin with a warm up which is weaker.

Proposition: If $f : R \to R$ is additive and continuous, then f is linear.

Linear and Additive functions

Lemma

If $f : R \rightarrow R$ is additive, then $f(cx) = cf(x)$ for all x and $c \in Q$.

Linear and Additive functions

Lemma

If $f : R \to R$ is additive, then $f(cx) = cf(x)$ for all x and $c \in Q$. In fancy words, f is a linear transformation when R is viewed as a Q vector space.

Linear and Additive functions

Lemma

If $f : R \to R$ is additive, then $f(cx) = cf(x)$ for all x and $c \in Q$. In fancy words, f is a linear transformation when R is viewed as a Q vector space.

Proof:

Linear and Additive functions

Lemma

If $f : R \to R$ is additive, then $f(cx) = cf(x)$ for all x and $c \in Q$. In fancy words, f is a linear transformation when R is viewed as a Q vector space.

Proof:
(1) implies by induction that for positive integer n, $f(nx) = nf(x)$.

Linear and Additive functions

Lemma

If $f : R \to R$ is additive, then $f(cx) = cf(x)$ for all x and $c \in Q$. In fancy words, f is a linear transformation when R is viewed as a Q vector space.

Proof:
(1) implies by induction that for positive integer n, $f(nx) = nf(x)$.
Therefore if n, m are positive integers, one has

$$f(m\frac{n}{m}x) = mf(\frac{n}{m}x)$$

Linear and Additive functions

Lemma

If $f : R \rightarrow R$ is additive, then $f(cx) = cf(x)$ for all x and $c \in Q$. In fancy words, f is a linear transformation when R is viewed as a Q vector space.

Proof:
(1) implies by induction that for positive integer n, $f(nx) = nf(x)$.
Therefore if n, m are positive integers, one has

$$f(m\frac{n}{m}x) = mf(\frac{n}{m}x)$$

giving

$$f(\frac{n}{m}x) = \frac{1}{m}f(nx) = \frac{n}{m}f(x).$$

Linear and Additive functions

Proposition: If $f : R \to R$ is additive and continuous, then f is linear.

Linear and Additive functions

Proposition: If $f : R \to R$ is additive and continuous, then f is linear.

Proof: Lemma gives $f(x) = xf(1)$ for all $x \in Q$.

Linear and Additive functions

Proposition: If $f : R \to R$ is additive and continuous, then f is linear.

Proof: Lemma gives $f(x) = xf(1)$ for all $x \in Q$. The two sides are continuous functions which agree on Q.

Linear and Additive functions

Proposition: If $f : R \to R$ is additive and continuous, then f is linear.

Proof: Lemma gives $f(x) = xf(1)$ for all $x \in Q$. The two sides are continuous functions which agree on Q. Since Q is dense, they must be equal for all x.

Linear and Additive functions

Proposition: If $f : R \to R$ is additive and continuous, then f is linear.

Proof: Lemma gives $f(x) = xf(1)$ for all $x \in Q$. The two sides are continuous functions which agree on Q. Since Q is dense, they must be equal for all x. Hence f is linear.
QED

Linear and Additive functions

Lemma

If f is additive but not linear, then f is unbounded in every interval about 0.

Linear and Additive functions

Lemma

If f is additive but not linear, then f is unbounded in every interval about 0. (I.e., f is additive and bounded in some interval about 0, then it is linear.)

Proof:
We assume that $f(1) = 1$.

Linear and Additive functions

Lemma

If f is additive but not linear, then f is unbounded in every interval about 0*.*
(I.e., f is additive and bounded in some interval about 0*, then it is linear.)*

Proof:
We assume that $f(1) = 1$. Fix the interval $(-\epsilon, \epsilon)$ around 0.

Linear and Additive functions

Lemma

If f is additive but not linear, then f is unbounded in every interval about 0. (I.e., f is additive and bounded in some interval about 0, then it is linear.)

Proof:
We assume that $f(1) = 1$. Fix the interval $(-\epsilon, \epsilon)$ around 0. If f is not linear, then there must exist positive $x \notin Q$ such that $f(x) \neq x$.

Linear and Additive functions

Lemma

If f is additive but not linear, then f is unbounded in every interval about 0. (I.e., f is additive and bounded in some interval about 0, then it is linear.)

Proof:

We assume that $f(1) = 1$. Fix the interval $(-\epsilon, \epsilon)$ around 0. If f is not linear, then there must exist positive $x \notin Q$ such that $f(x) \neq x$.

Kronecker's Theorem says that there exist integers n and m arbitrarily large so that

$$|nx - m| < \epsilon.$$

Linear and Additive functions

Lemma

If f is additive but not linear, then f is unbounded in every interval about 0. (I.e., f is additive and bounded in some interval about 0, then it is linear.)

Proof:
We assume that $f(1) = 1$. Fix the interval $(-\epsilon, \epsilon)$ around 0. If f is not linear, then there must exist positive $x \notin Q$ such that $f(x) \neq x$.
Kronecker's Theorem says that there exist integers n and m arbitrarily large so that

$$|nx - m| < \epsilon.$$

We then have

$$f(nx - m) = nf(x) - m = nx - m + n(f(x) - x).$$

Linear and Additive functions

Lemma

If f is additive but not linear, then f is unbounded in every interval about 0. (I.e., f is additive and bounded in some interval about 0, then it is linear.)

Proof:
We assume that $f(1) = 1$. Fix the interval $(-\epsilon, \epsilon)$ around 0. If f is not linear, then there must exist positive $x \notin Q$ such that $f(x) \neq x$.
Kronecker's Theorem says that there exist integers n and m arbitrarily large so that

$$|nx - m| < \epsilon.$$

We then have

$$f(nx - m) = nf(x) - m = nx - m + n(f(x) - x).$$

If n, m are very large and $|nx - m| < \epsilon$, then, since $f(x) - x \neq 0$,

Linear and Additive functions

Lemma

If f is additive but not linear, then f is unbounded in every interval about 0. (I.e., f is additive and bounded in some interval about 0, then it is linear.)

Proof:
We assume that $f(1) = 1$. Fix the interval $(-\epsilon, \epsilon)$ around 0. If f is not linear, then there must exist positive $x \notin Q$ such that $f(x) \neq x$.
Kronecker's Theorem says that there exist integers n and m arbitrarily large so that

$$|nx - m| < \epsilon.$$

We then have

$$f(nx - m) = nf(x) - m = nx - m + n(f(x) - x).$$

If n, m are very large and $|nx - m| < \epsilon$, then, since $f(x) - x \neq 0$, we have a point in $(-\epsilon, \epsilon)$ whose f value becomes in absolute value as large as we want.
QED

Linear and Additive functions

Theorem

If $f : R \to R$ is additive and measurable, then f is linear.

Linear and Additive functions

Theorem

If $f : R \to R$ is additive and measurable, then f is linear.

By lemma, enough to show that f is bounded in some interval $(-\epsilon, \epsilon)$ around 0.

Linear and Additive functions

Theorem

If $f : R \to R$ is additive and measurable, then f is linear.

By lemma, enough to show that f is bounded in some interval $(-\epsilon, \epsilon)$ around 0. Let $A_n = \{x : |f(x)| \leq n\}$.

Linear and Additive functions

Theorem

If $f : R \to R$ is additive and measurable, then f is linear.

By lemma, enough to show that f is bounded in some interval $(-\epsilon, \epsilon)$ around 0. Let $A_n = \{x : |f(x)| \leq n\}$. By our measurability assumption, the A_n's are measurable, clearly increasing and their union is R.

Linear and Additive functions

Theorem

If $f : R \to R$ is additive and measurable, then f is linear.

By lemma, enough to show that f is bounded in some interval $(-\epsilon, \epsilon)$ around 0. Let $A_n = \{x : |f(x)| \leq n\}$. By our measurability assumption, the A_n's are measurable, clearly increasing and their union is R. By continuity of measure from below, $m(A_{n_0}) > 0$ for some n_0.

Linear and Additive functions

Theorem

If $f : R \to R$ is additive and measurable, then f is linear.

By lemma, enough to show that f is bounded in some interval $(-\epsilon, \epsilon)$ around 0. Let $A_n = \{x : |f(x)| \leq n\}$. By our measurability assumption, the A_n's are measurable, clearly increasing and their union is R. By continuity of measure from below, $m(A_{n_0}) > 0$ for some n_0. By our exercise, $(-\epsilon, \epsilon) \subseteq A_{n_0} - A_{n_0}$ for some $\epsilon > 0$.

Linear and Additive functions

Theorem

If $f : R \to R$ is additive and measurable, then f is linear.

By lemma, enough to show that f is bounded in some interval $(-\epsilon, \epsilon)$ around 0. Let $A_n = \{x : |f(x)| \leq n\}$. By our measurability assumption, the A_n's are measurable, clearly increasing and their union is R. By continuity of measure from below, $m(A_{n_0}) > 0$ for some n_0. By our exercise, $(-\epsilon, \epsilon) \subseteq A_{n_0} - A_{n_0}$ for some $\epsilon > 0$. Then for all $x \in (-\epsilon, \epsilon)$, $x = a - b$ with $a, b \in A_{n_0}$ implying that

$$|f(x)| = |f(a-b)| = |f(a) - f(b)| \leq |f(a)| + |f(b)| \leq 2n_0.$$

Linear and Additive functions

Theorem

If $f : R \to R$ is additive and measurable, then f is linear.

By lemma, enough to show that f is bounded in some interval $(-\epsilon, \epsilon)$ around 0. Let $A_n = \{x : |f(x)| \leq n\}$. By our measurability assumption, the A_n's are measurable, clearly increasing and their union is R. By continuity of measure from below, $m(A_{n_0}) > 0$ for some n_0. By our exercise, $(-\epsilon, \epsilon) \subseteq A_{n_0} - A_{n_0}$ for some $\epsilon > 0$. Then for all $x \in (-\epsilon, \epsilon)$, $x = a - b$ with $a, b \in A_{n_0}$ implying that

$$|f(x)| = |f(a-b)| = |f(a) - f(b)| \leq |f(a)| + |f(b)| \leq 2n_0.$$

QED