Some Information regarding the Oral Exam.

How: The exam will take around 25-30 minutes and during that time you will be asked questions. Below you will find questions relating Riesz-Schauder Theory. During the exam you will be asked a few questions from the list. But you are expected to know the entire course; and you may be asked questions (and follow up questions) relating to anything in the course. However, the core of the exam will center around the questions below.

The questions.

Question 1: Sketch the argument (essentially Theorem 2.2 in *"the Dir and Neum. prob in gen domains."*) that if $f \in C(\partial D)$, D is a $C^{1,\alpha}-$domain, and

$$u(x) = \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} f(y) d\sigma(y),$$

then we may extend u to a function that is continuous on ∂D and for $x \in \partial D$

$$u(x) = \frac{f(x)}{2} + \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} f(y) d\sigma(y).$$

You do not have to learn the details; focus on how the integral is split up in different pieces and the principles that is used to estimate the pieces.

Question 2: Define a Hilbert space $\mathcal{H}$ and what it means for an operator $T : \mathcal{H} \mapsto \mathcal{H}$ to be bounded and linear.

Question 3: Define what it means for T^* to be the dual operator of an operator $T : \mathcal{H} \mapsto \mathcal{H}$. Say something about how to prove that

$$Tf(x) = \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} f(y) d\sigma(y)$$

and

$$T^* f(x) = \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_x} f(y) d\sigma(y)$$

are dual on $L^2(\partial D)$.

Question 4: Define what it means for an operator $T : \mathcal{B} \mapsto \mathcal{B}$ on a Banach space $\mathcal{B}$ to be compact.

Sketch an argument for the fact that if $T_j : \mathcal{B} \mapsto \mathcal{B}$ is a sequence of compact operators and $T_j \to T$, then T is compact.

Question 5: Formulate the Fredholm alternative in Hlbert spaces.

Question 6: Assuming that $T : \mathcal{B} \mapsto \mathcal{B}$ is a linear and compact operator on a Banach space and $S = I - T : \mathcal{B} \mapsto \mathcal{B}$. Use the fact that there exist a constant C_0 such that

$$\text{dist}(x, \mathcal{N}) \leq C_0 \|Sx\| \qquad \text{where } \mathcal{N} = \text{Kernel}(S)$$

to show that Range(S) is closed in $\mathcal{B}$.

Question 7: Assuming that $T : \mathcal{B} \mapsto \mathcal{B}$ is a linear and compact operator on a Banach space and $S = I - T : \mathcal{B} \mapsto \mathcal{B}$. Using the notation $S^j = \underbrace{S \circ S \circ ... \circ S}_{j \text{ times}}$ and $\mathcal{R}_j = \text{Range}(S^j)$.

Derive a contradiction from the assumption that, for every $j \in \mathbb{N}$, there exists an $x^j \in \mathcal{R}_j$, $\|x^j\| = 1$ and

$$\text{dist}(x^j, \mathcal{R}_j) \geq \frac{1}{2}.$$

Question 8: In several places in the course we use that $\frac{1}{2}I \pm T$ and $\frac{1}{2}I \pm T^*$ are related to solutions to the Laplace equation, for instance in Proposition 5.2 when we show that if $f \in \text{Kernel}(\frac{1}{2} - T^*)$ then

$$u(x) = \int_{\partial D} N(x, y) f(y) d\sigma(y)$$

is constant. Explain that argument.

Question 9: Given that

$$\text{Dim}(\text{Ker}(\frac{1}{2}I - T^*)) = \text{Dim}(\text{Ker}(\frac{1}{2}I - T)) = 1,$$

sketch a proof that

$$L^2(\partial D) = \text{Ker}(\frac{1}{2}I - T)^{\perp} \oplus \text{Ker}(\frac{1}{2}I - T^*).$$

Question 10: Given that

$$L^2(\partial D) = \text{Ker}(\frac{1}{2}I - T^*) \oplus \text{Range}(\frac{1}{2}I - T^*),$$

how can that be used to show wxistence of solutions to the Neumann problem in $C^{1,\alpha}-$domains?