Biomathematics:
A course on some applications of dynamical systems

Travelling waves

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Introduction

Travelling waves are usually associated with the one-dimensional wave equation

\[ c^{-2}u_{tt} = u_{xx} \]

for which the general solution is given by

\[ u(x, t) = f(x - ct) + g(x + ct). \]

The functions \( f \) and \( g \) are determined by the Cauchy-data, i.e., the initial conditions. For the diffusion equation

\[ u_t = u_{xx} + u, \]

no such travelling waves exist; if \( u(x, 0) = \phi(x) \), the solution is given by

\[ u(x, t) = e^t(K_t \ast \phi)(x), \quad K_t(x) = \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}. \]

However, it turns out that a “slight” modification of it can produce travelling waves:

\[ u_t = u_{xx} + f(u), \]

where \( f \) is nonlinear, such waves do quite often exist. In this chapter we will get a grip on when and in which direction such a wave travel. As a general observation, a wave of finite height will have to have a graph that connects one equilibrium of \( f(u) \) in \(-\infty\) to another one in \(+\infty\).

Introductory examples

We first discuss the simplest such case where

\[ f(u) = u(1 - u). \]

Consider the initial function below (black):

![Graph of u(x,t) with solutions for three different times in blue.](image)

The solution to the problem is shown for three different times in blue. What we see is that on each axis \( u(x, t) \to 1 \) as \( t \to \infty \), but also that these functions take on a very
specific form: there is a function $U_2(z)$ such that $u(x, t)$ converges to $U_2(x - 2t)$ on $x > 0$. The function $U_2$ is the unique solution of the problem

$$U'' + 2U + U(1 - U) = 0, \quad U(-\infty) = 1, \quad U(\infty) = 0.$$  

In fact, irrespective of $\phi$ (as long as it has bounded support) it will take on a very specific shape as $t \to \infty$.

If we instead take

$$f(u) = u(1 - u)(u - a), \quad 0 < a < 1,$$

we have two stable steady states, $u = 0$ and $u = 1$.

Hence

$$\begin{cases} 
0 \leq \phi < a \Rightarrow u(., t) \to 0, \\
a < \phi \leq 1 \Rightarrow u(., t) \to 1 \quad \text{as } t \to \infty.
\end{cases}$$

But what would the solution look like if we start out with an initial function which takes on values both below and above $a$?

We should have that $u(., t) \to 1$ or $u(., t) \to 0$ as $t \to \infty$. But which? It actually depends on which of the states $u = 1$ and $u = 0$ dominates, as measured by the sign of

$$\int_0^1 f(u) du.$$

In fact, what is true is that there is a unique $c$ such that the problem

$$U'' + cU' + f(U) = 0, \quad U(-\infty) = 1, \quad U(\infty) = 0,$$

has a unique solution $U_c$. Here $c$ has the same sign as the integral above and

$$u(x, t) \to U_c(x - ct), \quad x > 0 \text{ as } t \to \infty,$$

$$u(x, t) \to U_c(-x - ct), \quad x < 0 \text{ as } t \to \infty,$$

at least if $\phi > a$ on a sufficiently large interval. Hence we have two cases: if $c > 0$ we have a pair of diverging fronts and $u(., t) \to 1$ as $t \to \infty$, and if $c < 0$ they converge and $u(., t) \to 0$ as $t \to \infty$.

**Existence of travelling fronts**

The second order equation is equivalent to the first order system

$$u' = p, \quad p' = -cp - f(u)$$

and the vector field at the $u$-axes points downward, telling us that any trajectory joining the two critical points $(1, 0)$ and $(0, 0)$ must satisfy $p < 0$ everywhere. Thus we have that

$$U'(z) < 0 \text{ for all } z.$$

Because of this the curve $z \to (U(z), U'(z))$ can be represented by a graph

$$p = P(u), \quad 0 \leq u \leq 1,$$
with \( P(u) < 0 \) for \( 0 < u < 1 \). This function \( P(u) \) satisfies the equation

\[
P'(u) + \frac{f(u)}{P(u)} = -c, \quad P(0) = P(1) = 0.
\]

Obviously the converse is also true. Once we have found \( P(u) \) we can solve the equation

\[
U'(z) = P(U(z)), \quad U(0) = 1/2.
\]

Note that

\[
\int_{1/2}^{U(z)} \frac{du}{P(u)} = z,
\]

so we see that we get a monotone function defined for all \( z \) iff the integral of \( du/P(u) \) is divergent both in \( u = 0 \) and \( u = 1 \). It can be shown that this is the case. Also note that

\[
c \int_0^1 P(u) \, du = - \int_0^1 f(u) \, du,
\]

and since \( P < 0 \) we thus get that \( c \) has the same sign as \( \int_0^1 f(u) \, du \).

We now come to the question of existence of travelling waves. As we have seen, these are trajectories in \( p < 0 \) in the phase-space connecting \((1,0)\) and \((0,0)\), so some information about their existence can be deduced from studying these singular points. Let the singular point be \((a,0)\). Then the linearisation around \((a,0)\) has the matrix

\[
\begin{pmatrix}
0 & 1 \\
-f'(a) & -c
\end{pmatrix}
\]

which has eigenvalues

\[
\lambda_\pm = \lambda_\pm(c) = \frac{1}{2} (-c \pm \sqrt{c^2 - 4f'(a)})
\]

with corresponding eigenvectors

\[
e_\pm = (1, \lambda_\pm).
\]

If \( c^2 < 4f'(a) \), the eigenvalues are complex, which means that the trajectories to/from \((a,0)\) spiral around the point and thus cannot stay in \( p < 0 \). There are two more cases. If \( f'(a) > 0 \) and \( c^2 \geq 4f'(a) \) both eigenvalues have the same sign (depending on the sign of \( c \)) so either all trajectories leave \((a,0)\) (if \( c < 0 \)) or enter \((a,0)\) (if \( c > 0 \)) without a spiral behaviour. If \( f'(a) < 0 \) both eigenvalues are real but of opposite sign, so we have a saddle point in this case.

Ignoring the border line cases \( f'(a) = 0 \) we therefore have three possibilities:

\[\text{(I) Both (1,0) and (0,0) are saddle points, i.e.}\]

\[f'(0) < 0, \quad f'(1) < 0;\]

\[\text{(II) (1,0) is a saddle point and (0,0) a stable node, i.e.}\]

\[f'(0) > 0, \quad f'(1) < 0\]

in which case we must have \( c \geq 2\sqrt{f'(0)};\)
(III) \((1, 0)\) is an unstable node and \((0, 0)\) a saddle point, i.e.
\[
f'(0) < 0, \quad f'(1) > 0.
\]

But (III) is easily reduced to (II) by the simple transformation
\[
\bar{u} = 1 - u, \quad \bar{f}(\bar{u}) = -f(1 - \bar{u}),
\]
so we only have to consider (I) and (II). From now on we therefore assume \(u = 1\) is a saddle point and we need the outgoing trajectory from \((1, 0)\) to enter \((0, 0)\) without leaving the fourth quadrant.

In case (I) this means that this trajectory must be the same as the single ingoing trajectory to \((0, 0)\). Varying \(c\) it can be shown that there is a unique \(c\) for which this happens and we know that \(c\) then must have the same sign as \(\int_0^1 f(u) \, du\).

In case (II) it should be simpler. Here we need the outgoing trajectory from \((1, 0)\) to be “caught” by one orbit that goes into \((0, 0)\). Since all orbits close enough to the origin does so, this should not be too hard. It can be shown that this happens for all \(c\) above a lower limit.

**Remark** Consider a region
\[
D = \{(u, p); \ 0 < u < 1, \ -\rho(u) < p < 0, \ \rho(0) = \rho(1) = 0\}.
\]

If we can find a function \(\rho\) such that the vector \((u', p')\) points into the region everywhere, there must be such a trajectory. Such a region must contain the unstable trajectory from \((1, 0)\) (exercise). On \(p = 0\) we have \((u', p') = (0, -f(u))\) which obviously points into \(D\). A tangent vector to the lower boundary is given by \((1, -\rho'(u))\), so \((\rho'(u), 1)\) is an inward pointing normal to it, and the vector \((u', p')\) points into the region iff
\[
0 \leq (\rho'(u), 1) \cdot (u', p') = \rho'(u)p - cp - f(u) = -\rho'(u)\rho(u) + cp(u) - f(u),
\]
which is equivalent to
\[
c \geq \sup_{[0,1]} (\rho'(u) + \frac{f(u)}{\rho(u)}).
\]

**Example 1** For \(f(u) = u(1 - u)\) we have \(f'(0) = 1\) and \(f'(1) = -1\), so we must have \(c \geq 2\) for there to exist a trajectory. They do all in fact exist, but in computer simulations only the case \(c = 2\) occurs. The reason is that the speed of the wave depends on the shape of \(u_0\).

**Example 2** For \(f(u) = u(1 - u)(u - \alpha), \ \alpha \in (0, 1)\), we have \(f'(0) = -\alpha < 0\) and \(f'(1) = \alpha - 1 < 0\), so there is at most one unique travelling wave solution. For future reference we can note that in this case
\[
\int_0^1 f(u) \, du = \frac{1 - 2\alpha}{12}
\]
which is positive iff \(\alpha < 1/2\). In that case there is an explicit solution: the “Huxley
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In two (space) dimensions the corresponding equation would be

\[ u_t = \Delta u + f(u), \]

and a travelling wave starting in the origin would have the form \( u(x, t) = U(r - ct), \) where \( r = \sqrt{x_1^2 + x_2^2}. \) However, for a radial function \( u(x, y, t) = u(r, t) \) the original equation would be

\[ u_t = u_{rr} + \frac{1}{r} u_r + f(u), \]

so the equation for \( U \) should be

\[-cU' = U'' + \frac{1}{r} U' + f(U), \]

which does not work because of the middle term on the right hand side. But this disappear as \( r \to \infty, \) so asymptotically we could have something that looks a little like travelling waves also in the plane. The following graph shows the spread of the Black Death 1347-1352 in Europe.
Example 4 A simple model of the spread of an epidemic which might be particularly relevant to the spread of rabies by foxes is

\[
\begin{align*}
I_t &= D \Delta I + KIS - \mu I, \\
S_t &= -KIS,
\end{align*}
\]

where both $I$ and $S$ are population densities. The diffusion term is motivated by the fact that infected foxes often lose their territorial behaviour and invade ranges of neighbouring susceptibles. We assume an initial uniform spread $S_0$ of susceptibles and nondimensionalize:

\[
\begin{align*}
u(x,0) &\geq 0, \quad v(x,0) = 1.
\end{align*}
\]

This gives us the model (drop asterix)

\[
\begin{align*}
u_t &= \Delta u + uv - rv, \\
v_t &= -uv,
\end{align*}
\]

We want to see if it has a one-dimensional travelling wave solution $u(x,t) = f(x - t)$. 

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\[ v(x, t) = g(x - ct) \]

which solves

\[
\begin{aligned}
  f'' + cf + fg - rf &= 0, \\
  f(-\infty) &= f(\infty) = 0 \\
  g' &= fg/c, \\
  g(\infty) &= 1
\end{aligned}
\]

First we note that \( g \) is increasing, so that \( a = g(-\infty) \) exist and \( 0 < a < 1 \).

We can now prove that there exist travelling wave solutions iff \( r < 1 \). In such a case there is a unique wave for every \( c \geq c_0 = 2\sqrt{1-r} \) and we have that \( a \) is the solution of the equation \( a - r \ln a = 1 \). Similar to Fishers equation. Also, with localized outbreaks, the wave will travel with the lower speed \( c_0 \).

Exercises

**Exercise 1** For the budworm model with dispersion, compute the integral

\[
F(u; R) = \int_0^u f(t; R)dt.
\]

Which are its extreme points? Show that \( C'(R) > 0 \) and that when \( R \) is close to \( R_2 \) we have \( C(R) < 0 \) and when \( R \) is close to \( R_1 \) we have \( C(R) > 0 \).

**Exercise 2** Consider the model in the example above on the spread of rabies. Show that the system defining the travelling waves is equivalent to

\[
\begin{aligned}
  f' &= c(r \ln g - f' - g + 1) \\
  g' &= fg/c
\end{aligned}
\]

with the boundary conditions \( f(\infty) = 0, \ g(\infty) = 1 \). Determine the equilibria and their stability and conclude that a travelling wave solution can only occur for \( c \geq 2\sqrt{1-r} \).