FOURIER ANALYSIS & METHODS 2020.02.12

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Inhomogeneous or non-self adjoint boundary conditions

We wish to solve the homogeneous wave equation inside a rectangle:

$$\Box u = 0 \text{ inside a rectangle, } u(x,y,0) = f(x,y), \quad u_t(x,y,0) = 0,$$

$$u(x,y,t) = g(x,y) \text{ for } (x,y) \text{ on the boundary of the rectangle.}$$

We name this problem ♡. Here we have an inhomogeneous boundary condition. So, to solve the problem, we break it into two smaller problems which we tackle *one at a time:* divide and conquer.

Idea: deal with time independent boundary conditions by finding a steady state solution.

So, we begin by looking for

$$\Phi(x,y)$$

to satisfy

$$\Box \Phi = 0 \text{ inside the rectangle,}$$

$$\Phi = g \text{ on the boundary of the rectangle.}$$

Since the physical problem doesn't care where in space the rectangle is sitting, let us put it so that its vertices are at $(0,0)$, $(0,B)$, $(A,0)$, (A,B). Let us call this problem ♡♡.

Once we have found Φ, we will look for a solution w to solve

$$\Box w = 0 \text{ inside the rectangle,}$$

$$w(x,y,t) = 0 \text{ on the boundary of the rectangle,}$$

$$w(x,y,0) = f(x,y) - \Phi(x,y), \quad w_t(x,y,0) = 0.$$

Then, our solution to ♡ will be

$$u(x,y,t) = w(x,y,t) + \Phi(x,y).$$

So, we look for Φ to solve ♡♡.

Idea: deal with each inhomogeneous boundary component one at a time.

It is the same principle: divide and conquer. So, first, let us make nice zero boundary conditions on the sides, and just deal with the complicated boundary conditions on the top and bottom. Therefore we look for a function $\phi(x, y)$ which satisfies

$$\Box\phi = 0,$$
$$\phi(0, y) = \phi(A, y) = 0,$$
$$\phi(x, 0) = g(x, 0), \quad \phi(x, B) = g(x, B).$$

Idea: since the PDE is homogeneous and half of the BCs are good and homogeneous, use separation of variables.

We therefore write the PDE:

$$-X''Y - Y''X = 0 \implies -\frac{Y''}{Y} = \frac{X''}{X} = \lambda.$$

The BCs for X are $X(0) = X(A) = 0$. We have solved this problem. The solutions are, up to constant factors

$$X_n(x) = \sin\left(\frac{n\pi x}{A}\right), \quad \lambda_n = -\frac{n^2\pi^2}{A^2}.$$

The equation for the partner function is then:

$$-\frac{Y_n''}{Y_n} = \lambda_n \implies Y_n'' = \frac{n^2\pi^2}{A^2} Y_n.$$

A basis of solutions is given by real exponentials, or equivalently hyperbolic sines and cosines. Since our region contains 0, we have been given a hint that using the hyperbolic sines and cosines may be more simple. So, we follow that hint, with

$$Y_n(y) = a_n \cosh\left(\frac{n\pi y}{A}\right) + b_n \sinh\left(\frac{n\pi y}{A}\right).$$

Next we use superposition to create a super solution, which is legit because the PDE is homogeneous:

$$\phi(x, y) = \sum_{n\geq 1} X_n(x) Y_n(y).$$

To obtain the boundary conditions, we need

$$\phi(x, 0) = g(x, 0) = \sum_{n\geq 1} a_n X_n(x).$$

Hence, the coefficients

$$a_n = \frac{\langle g(x, 0), X_n\rangle}{||X_n||^2} = \frac{\int_0^A g(x, 0)\overline{X_n(x)}dx}{\int_0^A |X_n(x)|^2 dx}.$$

For the other BC, we need

$$\phi(x, B) = g(x, B) = \sum_{n\geq 1} X_n(x)\left(a_n \cosh\left(\frac{n\pi B}{A}\right) + b_n \sinh\left(\frac{n\pi B}{A}\right)\right).$$

Therefore we need

$$\left(a_n \cosh\left(\frac{n\pi B}{A}\right) + b_n \sinh\left(\frac{n\pi B}{A}\right)\right) = \frac{\langle g(x, B), X_n\rangle}{||X_n||^2}$$
$$= \frac{\int_0^A g(x, B) X_n(x) dx}{\int_0^A |X_n(x)|^2 dx}.$$

Solving for b_n we get

$$b_n = \frac{1}{\sinh(\frac{n\pi B}{A})} \left(\frac{\langle g(x,B), X_n \rangle}{||X_n||^2} - a_n \cosh\left(\frac{n\pi B}{A}\right) \right).$$

Next, we proceed similarly by searching for a function to fix up the BCs on the left and the right. Having dealt with the inhomogeneous BCs at the top and bottom, we set the BC there equal to zero. In that way, when we sum, we shall not mess up the function ϕ. So, we look for a solution to:

$$\Box\psi(x,y) = 0, \quad \psi(x,0) = \psi(x,B) = 0, \quad \psi(0,y) = g(0,y), \quad \psi(A,y) = g(A,y).$$

By symmetry, the solution will be given by

$$\sum_{n\geq 1} \widetilde{X_n}(y)\widetilde{Y_n(x)},$$

with

$$\widetilde{X_n(y)} = \sin\left(\frac{n\pi y}{B}\right),$$

and

$$\widetilde{Y_n(x)} = \widetilde{a_n}\cosh\left(\frac{n\pi x}{B}\right) + \widetilde{b_n}\sinh\left(\frac{n\pi x}{B}\right).$$

The coefficients come from the boundary conditions:

$$\widetilde{a_n} = \frac{\langle g(0,y), \widetilde{X_n} \rangle}{||\widetilde{X_n}||^2} = \frac{\int_0^B g(0,y)\widetilde{X_n(y)}dy}{\int_0^B |X_n(y)|^2 dy}.$$

The other one

$$\widetilde{b_n} = \frac{1}{\sinh\left(\frac{n\pi A}{B}\right)} \left(\frac{\langle g(A,y), \widetilde{X_n} \rangle}{||\widetilde{X_n}||^2} - \widetilde{a_n}\cosh\left(\frac{n\pi A}{B}\right) \right).$$

So, we have found

$$\psi(x,y) = \sum_{n\geq 1} \widetilde{X_n(y)}\widetilde{Y_n(x)}.$$

The full solution to this part of the problem is

$$\Phi(x,y) = \phi(x,y) + \psi(x,y).$$

Exercise 1. *Verify that this function satisfies both the PDE $\Box\Phi = 0$ as well as all of the boundary conditions.*

To complete the problem, we have only to solve the homogeneous wave equation with the lovely Dirichlet boundary condition and the initial condition with Φ subtracted. So, we are solving:

$$\Box u = 0, \quad u_t(x,y,0) = 0, \quad u(x,y,0) = f(x,y) - \Phi(x,y), \quad u = 0 \text{ on the boundary.}$$

> **Idea:** since we have homogeneous PDE and BC, use separation of variables and superposition.

We use separation of variables for t, x, and y. Write

$$u = TXY.$$

The PDE is

$$T''XY - X''TY - Y''TX = 0 \iff \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = \lambda.$$

Since we have nice homogeneous (Dirichlet) boundary conditions, we begin with the functions that depend on the position in the rectangle, that is X and Y.

Their equation is:

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda \implies \frac{X''}{X} = \lambda - \frac{Y''}{Y}.$$

OBS! The left and right sides depend on *different independent variables*. Hence, by the same reasoning that gave us λ, we get that

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y} = \mu.$$

Let us solve for X first.[1] So, we are looking to solve:

$$X'' = \mu X, \quad X(0) = X(A) = 0.$$

We have solved this before. The solutions are up to constant factors:

$$X_n(x) = \sin\left(\frac{n\pi x}{A}\right) \quad \mu_n = -\frac{n^2\pi^2}{A^2}.$$

This gives the equation for Y,

$$\frac{Y''}{Y} = \lambda - \mu_n, \quad Y(0) = Y(B) = 0.$$

Let us briefly call

$$\nu = \lambda - \mu_n.$$

Then, this is just the same equation but with different names for things:

$$Y'' = \nu Y, \quad Y(0) = Y(B) = 0.$$

Up to constant factors, the solutions are

$$Y_m(y) = \sin\left(\frac{m\pi y}{B}\right) \quad \nu_m = -\frac{m^2\pi^2}{B^2}.$$

Since

$$\nu_m = \lambda - \mu_n \implies \lambda = \lambda_{n,m} = \nu_m + \mu_n = -\frac{m^2\pi^2}{B^2} - \frac{n^2\pi^2}{A^2}.$$

Recalling the equation for the partner function, T, we have

$$T_{n,m}(t) = a_{n,m}\cos(\sqrt{|\lambda_{n,m}|}t) + b_{n,m}\sin(\sqrt{|\lambda_{n,m}|}t).$$

Hence we write

$$u(x,y,t) = \sum_{n,m\geq 1} T_{n,m}(t)X_n(x)Y_m(y).$$

The initial condition

$$u_t(x,y,0) = 0 \implies b_{n,m} = 0 \forall n,m.$$

The other condition is that

$$u(x,y,0) = f(x,y) - \Phi(x,y) = \sum_{n,m\geq 1} a_{n,m}X_n(x)Y_m(y).$$

Hence we require

$$a_{n,m} = \frac{\langle f - \Phi, X_nY_m\rangle}{||X_nY_m||^2} = \frac{\int_{[0,A]\times[0,B]}(f(x,y) - \Phi(x,y))X_n(x)Y_m(y)dxdy}{\int_{[0,A]\times[0,B]}|X_n(x)Y_m(y)|^2dxdy}.$$

[1] In this case, we could solve for either X or Y first, it actually does not matter which you choose.

The full solution is then

$$u(x, y, t) - \Phi(x, y).$$

Remark 1. *The eigenvalues of the two-dimensional SLP we solved above,*

$$\lambda_{n,m} = -\frac{m^2\pi^2}{B^2} - \frac{n^2\pi^2}{A^2}$$

are interesting to compare to the analogous one-dimensional case. In the analogous one dimension case, where we have

$$\mu_n = -\frac{n^2\pi^2}{A^2},$$

you can see that these are all square integer multiples of

$$\mu_1 = -\frac{\pi^2}{A^2}.$$

This is the mathematical reason that vibrating strings sound lovely. On the other hand, as long as the rectangle is not a square, that is $A \neq B$, it is no longer true that the $\lambda_{n,m}$ are all multiples of

$$\lambda_{1,1} = -\frac{\pi^2}{B^2} - \frac{\pi^2}{A^2}.$$

For this reason, vibrating rectangles can sound rather awful. You can listen to something along these lines (okay it's for tori not rectangles, but mathematically basically the same) here: `http://www.toroidalsnark.net/som.html`. Further exploration of the mathematics of music could make for an interesting bachelor's or master's thesis....

2. Heat equation example on an interval with an inhomogeneous boundary condition

We wish to solve the problem:

$$u_t - u_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$u(x, 0) = v(x),$$
$$u_x(4, t) = 0,$$
$$u(0, t) = 20.$$

Let us call this problem ♡. The boundary conditions are *not zero.* This will mean that the associated SLP does *not* have self-adjoint BCs, which is a big problem. We can use a similar "steady state" trick to deal with this. If the BC $u(0, t) = 20$ were instead $u(0, t) = 0$, then the BCs would be self adjoint BCs. So we want to make it so. Since the PDE is homogeneous, the

> **Idea:** Deal with non-self adjoint BCs which are independent of time by finding a steady state solution.

We want a function $f(x)$ which satisfies the equation

$$-f''(x) = 0,$$

and which gives us the bad BC

$$f(0) = 20.$$

We have a nice homogeneous BC on the other side, so we don't want to mess that up, so we want

$$f'(4) = 0.$$

Then, the function

$$f(x) = ax + b.$$

We use the BCs to compute

$$f(0) = 20 \implies b = 20.$$

$$f'(4) = 0 \implies a = 0.$$

Similar to before, if we add it to the solution of

$$u_t - u_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$u(x,0) = v(x),$$
$$u_x(4,t) = 0,$$
$$u(0,t) = 0.$$

it's going to screw up the IC. So, instead we look for the solution of

$$u_t - u_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$u(x,0) = v(x) - f(x),$$
$$u_x(4,t) = 0,$$
$$u(0,t) = 0.$$

This is now a PDE we know how to solve, so we call this problem ♡♡. We use SV to write $u = XT$ (just a means to an end).[2] Next, we get the equation

$$\frac{T'}{T} = \frac{X''}{X} = \lambda.$$

We solve the SLP

$$X'' = \lambda X, \quad X(0) = 0 = X'(4).$$

The reason we know this is an SLP satisfying the hypotheses of the theorem is because we verify that the BC is self-adjoint.

Exercise 2. *Verify that the only solutions for the cases $\lambda \geq 0$ are solutions which are identically zero.*

We only get $\lambda < 0$. Then, the solution is of the form

$$a_n \cos(\sqrt{|\lambda_n|}x) + b_n \sin(\sqrt{|\lambda_n|}x).$$

The BC at 0 tells us that

$$a_n = 0.$$

The BC at 4 tells us that

$$\cos(\sqrt{|\lambda_n|}4) = 0 \implies \sqrt{|\lambda_n|}4 = \frac{2n+1}{2}\pi \implies \sqrt{|\lambda_n|} = \frac{2n+1}{8}\pi.$$

[2]La fin justifie les moyens by M.C. Solaar is recommended listening.

We then also get

$$\lambda_n = -\frac{(2n+1)^2\pi^2}{64}.$$

We shall deal with the coefficients at the very end. So, we set

$$X_n(x) = \sin(\sqrt{|\lambda_n|}x).$$

The partner function

$$\frac{T_n'}{T_n} = \lambda_n \implies T_n(t) = \alpha_n e^{\lambda_n t} = \alpha_n e^{-(2n+1)^2\pi^2 t/64}.$$

We put it all together writing

$$u(x,t) = \sum_{n\ge 1} T_n(t)X_n(x).$$

To make the IC, we need

$$u(x,0) = \sum_{n\ge 1} T_n(0)X_n(x) = v(x) - f(x).$$

Since

$$T_n(0) = \alpha_n,$$

we need

$$\sum_{n\ge 1} \alpha_n X_n(x) = v(x) - f(x).$$

So we want the coefficients to be the Fourier coefficients of $v - f$, thus

$$\alpha_n = \frac{\langle v-f, X_n\rangle}{||X_n||^2} = \frac{\int_0^4 (v(x)-f(x))\overline{X_n(x)}dx}{\int_0^4 |X_n(x)|^2 dx}.$$

Our full solution is

$$U(x,t) = u(x,t) + f(x) = 20 + \sum_{n\ge 1} T_n(t)X_n(x).$$

2.1. **Exercises to solve oneself: hints.**

(1) (EO 25) Solve the problem:

$$u_{xx} + u_{yy} = y, \quad 0 < x < 2, \quad 0 < y < 1$$
$$u(x,0) = 0, \quad u(x,1) = 0$$
$$u(0,y) = y - y^3, \quad u(2,y) = 0.$$

Hint: divide and conquer. First, find a function which is independent of x to solve the inhomogeneous PDE. That is you want $f(y)$ to solve:

$$f''(y) = y, \quad f(0) = 0 = f(1).$$

Next solve the problem

$$v_{xx} + v_{yy} = 0,$$
$$v(x,0) = 0 = v(x,1),$$
$$v(0,y) = y - y^3 - f(y), \quad v(2,y) = -f(y).$$

Show that the solution to the original problem is given by

$$u(x,y) = v(x,y) + f(y).$$

(2) (EO 27) Solve the problem

$$u_{xx} + u_{yy} + 20u = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$
$$u(0, y) = u(1, y) = 0$$
$$u(x, 0) = 0, \quad u(x, 1) = x^2 - x.$$

Hint: divide and conquer. The PDE is homogeneous. So write $u = XY$. Plug it into the PDE. Solve for X function first because it has nice Dirichlet boundary conditions. Then solve for the partner Y function. Use the condition $u(x, 1) = x^2 - x$ to determine the unknown coefficients.

(3) (4.4:1) Solve the equation

$$u_{xx} + u_{yy} = 0$$

inside the square $0 \leq x, y \leq l$, subject to the boundary conditions:

$$u(x, 0) = u(0, y) = u(l, y) = 0, \quad u(x, l) = x(l - x).$$

Hint: follow the same procedure as the preceding exercise.

(4) (EO 3) Expand the function $\cos(x)$ in a sine series on the interval $(0, \pi/2)$. Use the result to compute

$$\sum_{n \geq 1} \frac{n^2}{(4n^2 - 1)^2}.$$

Hint: the coefficients in a sine series will be given by

$$\beta_n = \frac{4}{\pi} \int_0^{\pi/2} \cos(x) \sin(2nx) dx.$$

One way to make this integral easier is to expand stuff into complex exponentials from which you can obtain that

$$\cos(ax) \sin(bx) = \frac{1}{2} \left(\sin((a + b)x) - \sin((a - b)x) \right).$$

To compute the big sum at the end, use Parseval's equation.

(5) (4.2.2) Solve:

$$u_t = k u_{xx}, \quad u(x, 0) = f(x),$$
$$u(0, t) = C \neq 0, \quad u_x(l, t) = 0.$$

Hint: First deal with that icky inhomogeneous boundary condition $C \neq 0$ by finding a steady state solution as in lecture. This is $\phi(x)$ which has $\phi''(x) = 0$, $\phi(0) = C$, $\phi'(l) = 0$. Then, look for a solution to solve

$$u_t = k u_{xx}, \quad u(x, 0) = f(x) - \phi(x),$$
$$u(0, t) = 0, \quad u_x(l, t) = 0.$$

For this problem you can now use separation of variables, SLP theory, Hilbert space theory, and finally compute your coefficients using the initial data. It's all coming together!

(6) (4.3.1) Show that the function

$$b_n(t) := \frac{1}{n\pi c} \int_0^t \sin \frac{n\pi c(t - s)}{l} \beta_n(s) ds$$

solves the differential equation:

$$b_n''(t) + \frac{n^2 \pi^2 c^2}{l^2} b_n(t) = \beta_n(t),$$

as well as the initial conditions $b_n(0) = b_n'(0) = 0$. Hint: Use the fundamental theorem of calculus to compute $b_n'(t)$. Check the initial conditions by just substituting 0 for t. Next, compute the derivative of b_n' to get b_n'' and check the equation.

(7) (4.4.7) Solve the Dirichlet problem:

$$u_{xx} + u_{yy} = 0 \text{ in } S = \{(r, \theta) : 0 < r_0 \leq r \leq 1, \quad 0 \leq \theta \leq \beta\},$$
$$u(r_0, \theta) = u(1, \theta) = 0, \quad u(r, 0) = g(r), \quad u(r, \beta) = h(r).$$

Hint: Turn the equation into polar coordinates. An annulus looks like a rectangle in polar coordinates. Next separate variables and write $u = R(r)\Theta(\theta)$. Solve for the R function first because it has the beautiful boundary conditions. This is going to become an Euler equation like we solved for in lecture. So, check your lecture notes to see how we did that (Day 9). Once you have your R functions, they will be like $R_n(r)$, with corresponding λ_n, use this to find the partner Θ_n functions. Finally use g and h to determine the unknown coefficients. This part is a bit like finding the coefficients when solving the Dirichlet problem in a rectangle.

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).