FOURIER ANALYSIS & METHODS 2020.02.07

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. SLPs

Recall the definition of a regular SLP:

(1) a formally self-adjoint differential operator
$$L(f) = (rf')' + pf,$$
where r and p are real valued, r, r', and p are continuous, and $r > 0$ on $[a, b]$.

(2) self-adjoint boundary conditions:
$$B_i(f) = \alpha_i f(a) + \alpha_i' f'(a) + \beta_i f(b) + \beta_i' f'(b) = 0, \quad i = 1, 2.$$
The self adjoint condition further requires that the coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i'$ are such that for all f and g which satisfy these conditions
$$r(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

(3) a positive, continuous function w on $[a, b]$.

The SLP is to find all solutions to the BVP
$$L(f) + \lambda w f = 0, \quad B_i(f) = 0, \quad i = 1, 2.$$
The eigenvalues are all numbers λ for which there exists a corresponding non-zero eigenfunction f so that together they satisfy the above equation, and f satisfies the boundary condition. The magical theorem about SLPs says that for such a regular SLP, there exists solutions $\{\phi_n\}_{n\geq 1}$ with corresponding eigenvalues λ_n such that these $\{\phi_n\}_{n\geq 1}$ are an orthogonal basis for the weighted $\mathcal{L}^2$ space, $\mathcal{L}^2_w(a, b)$. Moreover, these eigenvalues are all *real.* Let's see just what makes this theorem so magical...

1.1. SLP example for a PDE.

Here is how the SLP theory can be useful in practice. We are given the problem
$$u_t - u_{xx} = 0, \quad u_x(0, t) = \alpha u(0, t), \quad u_x(l, t) = -\alpha u(l, t), \quad u(x, 0) = f(x).$$
Above, we assume that
$$\alpha > 0, \quad f \in \mathcal{L}^2.$$
These boundary conditions are based on Newton's law of cooling: the temperature gradient across the ends is proportional to the temperature difference between the

ends and the surrounding medium. It is a homogeneous PDE, so we have good chances of being able to solve it using separation of variables. Thus, we write

$$u(x,t) = X(x)T(t) \implies T'(t)X(x) - X''(x)T(t) = 0 \implies \frac{T'}{T} = \frac{X''}{X}.$$

This means both sides are equal to a constant. Call it λ. We start with the x side, because we have more information about that due to the BCs. Are they self-adjoint BCs? Let's check! In the definition of SLP, we are looking for X to satisfy

$$\frac{X''}{X} = \lambda \iff X'' = \lambda X \iff X'' - \lambda X = 0.$$

OBS! The relationship between the constant we have named λ from the PDE has the *opposite* sign as the corresponding term in an SLP. So, the SLP would look like

$$X'' + \Lambda X = 0 \quad \Lambda = -\lambda.$$

The r and w are both 1 in the definition of SLP, and the p is 0. The $a = 0$ and $b = l$. So, we need to check that if f and g satisfy

$$f'(0) = \alpha f(0), \quad g'(l) = -\alpha g(l)$$

then

$$(\bar{g}f' - \bar{g}'f)\big|_0^l = 0.$$

We plug it in

$$\bar{g}(l)f'(l) - \bar{g}'(l)f(l) - \bar{g}(0)f'(0) + \bar{g}'(0)f(0)$$
$$= -\bar{g}(l)\alpha f(l) + \alpha \bar{g}(l)f(l) - \bar{g}(0)\alpha f(0) + \alpha \bar{g}(0)f(0) = 0.$$

Yes, the BC is a self-adjoint BC. So, the SLP theorem says there exists an $\mathcal{L}^2$ OB of eigenfunctions. What are they? We check the cases.

$$X'' = \lambda X.$$

What if $\lambda = 0$? Then

$$X(x) = ax + b.$$

To get

$$X'(0) = \alpha X(0) \implies a = \alpha b \implies b = \frac{a}{\alpha}.$$

Next,

$$X'(l) = -\alpha X(l) \implies a = -\alpha\left(al + \frac{a}{\alpha}\right) = -a(\alpha l + 1).$$

Presumably $a \neq 0$ because if $a = 0$ the whole solution is just 0. So, we can divide by it and we get

$$\implies 1 = -(\alpha l + 1) \implies \alpha l = -2.$$

Since $l > 0$ and $\alpha > 0$, this is impossible. So, no non-zero solutions for $\lambda = 0$.

Next we try $\lambda > 0$. Then the solution looks like

$$X(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}$$

or equivalently, we can use sinh and cosh, to write

$$X(x) = a\cosh(\sqrt{\lambda}x) + b\sinh(\sqrt{\lambda}x).$$

We try out the BCs. They require

$$X'(0) = \alpha X(0) \iff a\sqrt{\lambda}\sinh(0) + b\sqrt{\lambda}\cosh(0) = \alpha\left(a\cosh(0) + b\sinh(0)\right)$$
$$\iff b\sqrt{\lambda} = \alpha a \implies b = \frac{\alpha a}{\sqrt{\lambda}}.$$

We check out the other BC:

$$X'(l) = -\alpha X(l) \iff a\sqrt{\lambda}\sinh(\sqrt{\lambda}l) + \alpha a\cosh(\sqrt{\lambda}l) = -\alpha\left(a\cosh(\sqrt{\lambda}l) + \frac{\alpha a}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}l)\right).$$

$$\iff a\sqrt{\lambda}\sinh(\sqrt{\lambda}l) + \frac{\alpha^2 a}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}l) = -2\alpha a\cosh(\sqrt{\lambda}l)$$

If $a = 0$ the whole solution is zero, so we presume that is not the case and divide by a. Then this requires

$$\frac{\sinh(\sqrt{\lambda}l)}{\cosh(\sqrt{\lambda}l)} = \frac{-2\alpha}{\sqrt{\lambda} + \alpha^2/\sqrt{\lambda}}.$$

The left side is positive, but the right side is negative. ↯

Thus, we finally try $\lambda < 0$. Then the solution looks like

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To get

$$X'(0) = \alpha X(0) \implies b\sqrt{|\lambda|} = \alpha a \implies b = \frac{\alpha a}{\sqrt{|\lambda|}}.$$

Next we need

$$X'(l) = -\alpha X(l)$$

$$\implies -a\sqrt{|\lambda|}\sin(\sqrt{|\lambda|}l) + \frac{\alpha a}{\sqrt{|\lambda|}}\sqrt{|\lambda|}\cos(\sqrt{|\lambda|}l) = -\alpha\left(a\cos(\sqrt{|\lambda|}l) + \frac{\alpha a}{\sqrt{|\lambda|}}\sin(\sqrt{|\lambda|}l)\right).$$

Presumably $a \neq 0$ because if that is the case then the whole solution is 0. So, we may divide by a, and we need

$$2\alpha\cos\sqrt{|\lambda|} = \sin(\sqrt{|\lambda|}l)\left(\sqrt{|\lambda|} - \frac{\alpha^2}{\sqrt{|\lambda|}}\right).$$

This is equivalent to

$$\frac{2\alpha}{\sqrt{|\lambda|} - \frac{\alpha^2}{\sqrt{|\lambda|}}} = \tan(\sqrt{|\lambda|}l)$$

$$\iff \frac{2\alpha\sqrt{|\lambda|}}{|\lambda| - \alpha^2} = \tan(\sqrt{|\lambda|}l).$$

Well, that's pretty weird, but according to the SLP theory, the sequence

$$\{\lambda_n\}_{n\geq 1} \text{ and } \{X_n(x)\}_{n\geq 1}, \quad X_n(x) = a_n\left(\cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}}\sin(\sqrt{|\lambda_n|}x)\right)$$

of eigenvalues and corresponding eigenfunctions is an orthogonal basis of $\mathcal{L}^2$. Here since we are solving a PDE, it is most convenient to leave the coefficients simply as a_n and solve for them according to the initial conditions of the PDE.

The partner functions

$$T_n(t) \text{ satisfy } T_n'(t) = \lambda_n T_n(t) \implies T_n(t) = e^{\lambda_n t}.$$

Here it is good to note that the $\lambda_n < 0$ and tend to $-\infty$ as $n \to \infty$ which follows from the Adult Spectral Theorem, because in the SLP terminology,

$$\Lambda_n = -\lambda_n \to \infty \implies \lambda_n \to -\infty.$$

So, for heat, that is realistic. We build the solution using superposition because the PDE is linear and homogeneous, so

$$u(x,t) = \sum_{n \geq 1} T_n(t) X_n(x).$$

Since we wish this to be equal to the initial data at $t = 0$, we demand

$$u(x,0) = \sum_{n \geq 1} a_n \left(\cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x) \right) = f(x).$$

By the SLP theory, the functions above form an OB, so we can expand our initial data function in terms of this OB. To do this we compute

$$a_n = \frac{\langle f(x), \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x) \rangle}{|| \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x) ||^2},$$

where

$$\langle f(x), \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x) \rangle = \int_0^l f(x) \overline{(\cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x))} dx,$$

$$|| \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x) ||^2 = \int_0^l | \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x) |^2 dx.$$

1.2. **SLP example.** SLPs may come from solving a PDE, but to avoid overcomplicating things, sometimes you will just need to solve an SLP by itself. For example:

$$(xf')' + \lambda x^{-1} f = 0, \quad f(1) = f(b) = 0, \quad b > 1.$$

In this example the function $r(x) = x$, and the function $p(x) = 0$, whilst the weight function $w(x) = x^{-1}$. Let us consider three cases for λ.

Case $\lambda = 0$**:** If $\lambda = 0$, then the equation becomes

$$xf'' + f' = 0,$$

which we can re-arrange to

$$\frac{f''}{f'} = -\frac{1}{x}.$$

The left side is the derivative of $\log(f')$. So, integrating both sides (saving the constant for later):

$$\log(f') = -\log(x).$$

Exponentiating both sides we get

$$f' = \frac{1}{x} \implies f(x) = A \log(x) + B,$$

for some constants A and B. The boundary conditions demand that

$$f(1) = 0 \implies B = 0.$$

The other boundary condition demands that

$$f(b) = 0 \implies A = 0, \text{ since } b > 1 \text{ so } \log(b) > 0.$$

We are left with the zero function. That is never an eigenfunction. So $\lambda = 0$ is not an eigenvalue for this SLP.

Case $\lambda > 0$**:** If $\lambda > 0$, we observe that the equation we have is something called an Euler equation. (Or we look up the ODE section of Beta and search for this type of ODE, and see that Beta tells us this is an Euler equaiton). Consequently, we look for solutions of the form

$$f(x) = x^{\nu}.$$

The differential equation we wish to solve is:

$$xf'' + f' + \lambda x^{-1} f = 0 \implies x^2 f'' + xf' + \lambda f = 0,$$

so substituting $f(x) = x^{\nu}$, this becomes

$$x^2(\nu)(\nu - 1)x^{\nu-2} + x\nu x^{\nu-1} + \lambda x^{\nu} = 0.$$

This simplifies to:

$$x^{\nu}\left(\nu^2 - \nu + \nu + \lambda\right) = 0 \implies \nu^2 = -\lambda.$$

Since $\lambda > 0$, this means

$$\nu = \pm i\sqrt{\lambda}.$$

So, a basis of solutions is $x^{i\sqrt{|\lambda|}}$ and $x^{-i\sqrt{\lambda}}$. Note that

$$x^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda}\log(x)}.$$

By Euler's formula, an equivalent basis of solutions is

$$\cos(\sqrt{\lambda}\log(x)), \quad \sin(\sqrt{\lambda}\log(x)).$$

Hence in this case our solution is of the form:

$$f(x) = A\cos(\sqrt{\lambda}\log(x)) + B\sin(\sqrt{\lambda}\log(x)).$$

The boundary conditions demand that

$$f(1) = 0 \implies A = 0.$$

The second boundary condition demands that

$$B\sin(\sqrt{\lambda}\log(b)) = 0.$$

Since we do not seek the zero function, we presume that $B \neq 0$ and thus require

$$\sin(\sqrt{\lambda}\log(b)) = 0 \implies \sqrt{\lambda}\log(b) = n\pi, \quad n \in \mathbb{N}.$$

We therefore have countably many eigenfunctions and eigenvalues, which we may index by the natural numbers, writing

$$\lambda_n = \frac{n^2\pi^2}{(\log b)^2}, \quad f_n(x) = \sin\left(\frac{n\pi\log(x)}{\log(b)}\right).$$

Nice.

The last case to consider is **case** $\lambda < 0$**:** We proceed similarly as above and obtain that a basis of solutions is

$$x^{\pm\sqrt{|\lambda|}}.$$

Write our solution as

$$f(x) = Ax^{\sqrt{|\lambda|}} + Bx^{-\sqrt{|\lambda|}}.$$

The boundary conditions demand that:

$$f(1) = 0 \implies A + B = 0 \implies B = -A.$$

The next boundary condition demands that:

$$f(b) = Ab^{\sqrt{|\lambda|}} - Ab^{-\sqrt{|\lambda|}} = 0 \implies A = 0 \text{ or } b^{\sqrt{|\lambda|}} = b^{-\sqrt{|\lambda|}} \implies b^{2\sqrt{|\lambda|}} = 1 \implies \sqrt{|\lambda|} = 0 \lightning.$$

Thus the only way for the boundary conditions to be satisfied is if the eigenfunction is the zero function, but this is not an eigenfunction! Hence no negative λ solutions.

The magical SLP theorem tells us that these rather peculiar functions

$$\{f_n(x)\}_{n\geq 1}$$

are an orthogonal basis for $\mathcal{L}^2_{1/x}(1,b)$. This means that for any $g \in \mathcal{L}^2_{1/x}(1,b)$, we can expand it as a Fourier series with respect to this basis. The coefficients will be

$$\frac{\langle g, f_n\rangle_{1/x}}{||f_n||^2_{1/x}}, \quad \langle g, f_n\rangle_{1/x} = \int_1^b g(x)\overline{f_n(x)}x^{-1}dx, \quad ||f_n||^2_{1/x} = \int_1^b |f_n(x)|^2 x^{-1}dx.$$

If the function we wish to expand is specified, we could compute these integrals.

1.3. **Another SLP example.** Consider the problem

$$(x^2 f')' + \lambda f = 0, \quad f(1) = f(b) = 0, \quad b > 1.$$

Here we have $r(x) = x^2$ and $w(x) = 1$. The equation is:

$$2xf' + x^2 f'' + \lambda f = 0.$$

We shall consider the three cases for λ.

Case $\lambda = 0$: In this case the equation simplifies to

$$x^2 f'' + 2xf' = 0 \implies \frac{f''}{f'} = -\frac{2}{x} \implies (\log(f'))' = -\frac{2}{x} \implies \log(f') = -2\log x \implies f' = e^{-2\log x} = x^{-2}.$$

So, this gives us a solution of the form

$$f(x) = -A\frac{1}{x} + B.$$

Let us verify the boundary conditions. We require $f(1) = 0$ so this means

$$-A + B = 0 \implies B = A.$$

We also require $f(b) = 0$ so this means

$$-A\frac{1}{b} + B = 0 = \frac{-A}{b} + A \implies \frac{A}{b} = A \implies b = 1 \text{ or } A = 0.$$

So since $b > 1$ the only solution here is the zero function which is not an eigenfunction.

Case $\lambda > 0$: We consider the fact that this is an Euler equation, so we look for solutions of the form $f(x) = x^\nu$. Then the equation looks like:

$$x^2(\nu)(\nu-1)x^{\nu-2} + 2x(\nu)x^{\nu-1} + \lambda x^\nu = 0 \iff x^\nu\left(\nu^2 - \nu + 2\nu + \lambda\right) = 0$$

so we need ν to satisfy:

$$\nu^2 + \nu + \lambda = 0.$$

This is a quadratic equation, so we solve it:

$$\nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

So, actually the cases $\lambda > 0$ and $\lambda < 0$ really should split up into whether $\lambda = \frac{1}{4}$ or is larger or smaller. If $\lambda = \frac{1}{4}$, then we are only getting one solution this way, $x^{-1/2}$. To get a second solution we multiply by $\log x$.

Exercise 1. *Plug the function $x^{-1/2}\log x$ into the SLP for the value $\lambda = \frac{1}{4}$. Verify that it satisfy the equation.*

Now, let's see if such a function will satisfy the boundary conditions. We need

$$Ax^{-1/2} + Bx^{-1/2}\log(x)\Big|_{x=1} = 0 \implies A = 0.$$

We also need

$$Bb^{-1/2}\log(b) = 0, \quad b > 1 \implies B = 0.$$

So we only get the zero solution in this case.

When $\lambda < \frac{1}{4}$, solutions are of the form

$$Ax^{\nu_+} + Bx^{\nu_-}, \quad \nu_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

Exercise 2. *Check the boundary conditions. Verify that they are satisfied if and only if $A = B = 0$.*

Finally we consider $\lambda > \frac{1}{4}$. Then we have

$$\nu_\pm = -\frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}} \implies f(x) = \frac{A}{\sqrt{x}}x^{i\sqrt{\lambda-1/4}} + \frac{B}{\sqrt{x}}x^{-i\sqrt{\lambda-1/4}}.$$

Using Euler's formula, this is equivalently expressed as

$$\frac{\alpha}{\sqrt{x}}\cos(\sqrt{\lambda - 1/4}\log x) + \frac{\beta}{\sqrt{x}}\sin(\sqrt{\lambda - 1/4}\log x).$$

Due to the boundary condition at $x = 1$ we must have $\alpha = 0$. So to obtain the other boundary condition, we need

$$\sin(\sqrt{\lambda - 1/4}\log b) = 0 \implies \sqrt{\lambda - 1/4}\log b = n\pi, \quad n \in \mathbb{N}.$$

Hence

$$\lambda = \lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\log b)^2}, \quad f_n(x) = x^{-1/2}\sin\left(\frac{n\pi\log x}{\log b}\right).$$

Note that in general we are not bothering to normalize our eigenfunctions because it is rather tedious and not fundamental to our learning experience in this subject.

1.4. **Exercises for the week: Answers.** Those exercises from [1] which one should solve are:

(1) (3.3.1) Show that if $\{f_n\}_{n\geq 1}$ are elements of a Hilbert space, H, and we have for some $f \in H$ that

$$\lim_{n\to\infty} f_n = f,$$

then for all $g \in H$ we have

$$\lim_{n\to\infty}\langle f_n, g\rangle = \langle f, g\rangle.$$

Answer: we would like to prove

$$\lim_{n\to\infty}\langle f_n, g\rangle = \langle f, g\rangle.$$

This is equivalent to proving

$$\lim_{n\to\infty}\langle f_n, g\rangle - \langle f, g\rangle = 0.$$

So, next we follow the hint and estimate

$$|\langle f_n, g\rangle - \langle f, g\rangle| = |\langle (f_n - f), g\rangle| \leq ||f_n - f||||g||.$$

The meaning of

$$\lim_{n\to\infty} f_n = f$$

in a Hilbert space is that

$$\lim_{n\to\infty} ||f_n - f|| = 0.$$

Hence, by some theorem about the product of limits, as long as they exist (obs! $\lim_{n\to\infty} ||g|| = ||g||$, it's just not changing at all), we have

$$\lim_{n\to\infty} ||f_n - f|| ||g|| = ||g|| \lim_{n\to\infty} ||f_n - f|| = 0.$$

(2) (3.3.2) Show that for all f, g in a Hilbert space one has

$$|||f|| - ||g||| \le ||f-g||.$$

Answer: We follow the hint. For any real numbers a and b,

$$|a-b|^2 = a^2 - 2ab + b^2.$$

Next, we apply this fact with $a = ||f||$ and $b = ||g||$ to obtain that

$$|||f|| - ||g|||^2 = ||f||^2 - 2||f|| ||g|| + ||g||^2.$$

We compare this to

$$||f-g||^2 = ||f||^2 - 2\Re\langle f, g\rangle + ||g||^2,$$

since

$$||f|| ||g|| \ge \Re\langle f, g\rangle \implies ||f||^2 - 2\Re\langle f, g\rangle + ||g||^2 \ge ||f||^2 - 2||f|| ||g|| + ||g||^2.$$

Thus we obtain

$$||f-g||^2 \ge |||f|| - ||g|||^2.$$

Taking the square root of both sides completes the proof.

(3) (3.3.10.d) Use Parseval's equation to compute

$$\sum_{n\ge 1} \frac{\sin^2(na)}{n^4}.$$

Answer:

$$\frac{a^2(\pi - a)^2}{6}.$$

(4) (3.4.1) Show that $\{e^{2\pi i(mx+ny)}\}_{n,m\in\mathbb{Z}}$ is an orthogonal set in $\mathcal{L}^2(R)$ where R is any square whose sides have length one and are parallel to the coordinate axes. Answer:

$$\int_{x=a}^{a+1} \int_{y=b}^{b+1} e^{2\pi i(mx+ny)} e^{-2\pi i(kx+\ell y)} dx dy = \int_{x=a}^{a+1} e^{2\pi i(m-k)x} dx \int_{y=b}^{b+1} e^{2\pi i(n-\ell)y} dy.$$

For $m \neq k$,

$$\int_{x=a}^{a+1} e^{2\pi i(m-k)x} dx = \left. \frac{e^{2\pi i(m-k)x}}{2\pi i(m-k)} \right|_a^{a+1}.$$

The function above is 1 periodic, so this is zero. Same holds for $n \neq \ell$.

(5) (3.4.6) Find an example of a sequence $\{f_n\}$ in $\mathcal{L}^2(0,\infty)$ such that $f_n(x) \to 0$ uniformly for all $x > 0$ but $f_n \not\to 0$ in the $\mathcal{L}^2$ norm. Answer: let

$$f_n(x) := \begin{cases} \frac{1}{\sqrt{x+\sqrt{n}}} & 0 \le x \le n \\ 0 & x > n \end{cases}.$$

Then

$$0 \le \lim_{n\to\infty} f_n(x) \le \lim_{n\to\infty} \frac{1}{\sqrt{\sqrt{n}}} = 0.$$

So the convergence to zero is uniform on $[0,\infty)$. On the other hand

$$||f_n||_{\mathcal{L}^2}^2 = \int_0^\infty |f_n(x)|^2 dx = \int_0^n \frac{1}{x+\sqrt{n}} dx = \ln(x+\sqrt{n})\big|_{x=0}^n$$
$$= \ln(n+\sqrt{n}) - \ln(\sqrt{n}) = \ln\left(\frac{n+\sqrt{n}}{\sqrt{n}}\right) = \ln(\sqrt{n}+1).$$

This simultaneously shows that $f_n \in \mathcal{L}^2(0,\infty)$ for all n, as well as that the $\mathcal{L}^2$ norm of f_n tends to infinity.

(6) (3.5.7) Find all solutions f on $[0,1]$ and all corresponding λ to the equation:

$$f'' + \lambda f = 0, \quad f(0) = 0, \quad f'(1) = -f(1).$$

Answer: the eigenvalues are $\lambda_n = \nu_n^2$ where ν_n are the positive solutions of $\tan(\nu) = -\nu$, and the eigenfunctions are $\sin(\nu_n x)$.

(7) (4.2.3) Let $f(x)$ be the initial temperature at the point x in a rod of length ℓ, mathematicized as the interval $[0,\ell]$. Assume that heat is supplied at a constant rate at the right end, in particular $u_x(\ell,t) = A$ for a constant value A, and that the left end is held at the constant temperature 0, so that $u(0,t) = 0$. Find a series expansion for the temperature $u(x,t)$ such that the initial temperature is given by $f(x)$. Answer:

$$u(x,t) = Ax + \sum_{n\ge1}\left(b_n + \frac{(-1)^n 8A\ell}{(2n-1)^2\pi^2}\right) e^{-(n-1/2)^2\pi^2 kt/(\ell^2)} \sin((n-1/2)\pi x/\ell).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).