TMA947 / MMG621 — Nonlinear optimization

Exercise 2 – Introduction to optimality conditions and Unconstrained optimization

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E2.1 (easy) Weierstrass theorem claims that an optimal solution of a continuous objective function always exists if either the feasible set is closed and bounded (compact) or if the set is closed and the objective coercive (i.e. $\lim_{\|\boldsymbol{x}\|\to\infty} f(\boldsymbol{x}) = \infty$ in a minimization problem). Consider the following problems. Does an optimal solution exist? If not, why does the problem violate Weierstrass theorem? (Hint: draw the problems)

(a)

$$\begin{array}{ll} \text{minimize} & x^2 \\ \text{subject to} & 0 < x \leq 1 \end{array}$$

(b)

$$\begin{array}{ll} \text{minimize} & e^x \\ \text{subject to} & x \leq 0 \end{array}$$

(c)

$$\begin{array}{ll} \text{minimize} & x^3 - 64x \\ \text{subject to} & |x| < 10 \end{array}$$

(d)

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & |x| \leq 3 \end{array}$$

where

$$f(x) = \begin{cases} -x^2 & \text{if } |x| < 2, \\ -x & \text{if } |x| \geq 2. \end{cases}$$

E2.2 (easy) Consider the following problems. Does an optimal solution exist? If not, why do the problems violate Weierstrass theorem? (Hint: draw the feasible region of the problems)

(a)

$$\begin{aligned} \text{minimize} \quad & -x - y, \\ \text{subject to} \quad & 0 \geq x, \\ & 0 \geq y, \\ & x^2 + y^2 < 1. \end{aligned}$$

(b)

$$\begin{aligned} \text{minimize} \quad & -x - y, \\ \text{subject to} \quad & x + y \geq 1, \\ & 3x \geq y, \\ & y \geq 2x. \end{aligned}$$

E2.3 (medium) Construct a continuous objective f such that no optimal solution exists to the problem $\min_{\boldsymbol{x} \in X} f(\boldsymbol{x})$. Comment on how Weierstrass is violated.

(a)

$$X = \{x \in \mathbb{R} \mid 0 \leq x < 10 \text{ or } 11 \leq x \leq 12\},$$

(b)

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid 2^2 x_1^2 + 5^2 x_2^2 \leq 10^2\} \setminus \{(1,1)\}.$$

E2.4 (easy) State the set of feasible directions for the feasible set X at the point $\boldsymbol{x}$. (Hint: draw the feasible set)

(a)

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid 3x_1^2 + x_2^3 \leq 1\}, \boldsymbol{x} = (0, 0.99)^T.$$

(b)

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}, \boldsymbol{x} = (1/\sqrt{2}, 1/\sqrt{2})^T.$$

(c)

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (x_1 - 1)^2 + x_2^2 \geq 1, (x_1 + 1)^2 + x_2^2 \geq 1\}, \boldsymbol{x} = (0, 0)^T.$$

(d)

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid 2x_1 + 3x_2 = 5\}, \boldsymbol{x} = (1, 1)^T.$$

(e)

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 = x_2^2\}, \boldsymbol{x} = (1, 1)^T.$$

E2.5 (easy) Is the point $(0,1,0)^T$ an optimal solution to the problem

$$\begin{array}{ll} \text{minimize} & (x+1)^2 + (y-2)^2 + (z+1)^2, \\ \text{subject to} & x^2 + y^2 \leq 1, \\ & \quad x \geq 0, \\ & \quad y \geq 0, \\ & \quad z \geq 0. \end{array}$$

Use Theorem 4.23.

E2.6 (easy) Solve the unconstrained problem

$$\text{minimize } \boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x},$$

by using the optimality conditions for unconstrained optimization if

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

E2.7 (easy) An investor is seeking to choose a portfolio into a set of stocks wisely. To do this s/he has collected historical data of mean return and covariance of the set of stocks. Assume that the available stocks to invest in is denoted by $\mathcal{I}$. For each $i \in \mathcal{I}$, we assume that the historical mean rate of return when investing 1 SEK is r_i. Further, for each pair $i, j \in \mathcal{I}$, let q_{ij} denote the historical covariance of the rate of return when investing 1 SEK into both stocks i and j (note that $q_{ii} > 0$ denotes the variance of investing 1 SEK in the stock i). Finally assume that the investor has M SEK of money available for investment.
Formulate an optimization model for minimizing the risk of investment (measured as the covariance of the return of the portfolio), subject to the constraint that the expected return of the portfolio is at least R_0, for some $R_0 > 0$.
Under what conditions can you guarantee that your model has an optimal solution? Under what conditions can you guarantee that it has optimal solution if we assume that the investor has an infinite amount of money available for investment (i.e., $M = \infty$)?

E2.8 (easy) Consider the following problem

$$\text{minimize } \boldsymbol{x}^T A \boldsymbol{x},$$

where $A = \begin{pmatrix} 0.5 & 2 \\ 0 & 0.5 \end{pmatrix}$. Let $\boldsymbol{x}^0 = (0, -1)^T$.

(a) Starting at $\boldsymbol{x}^0$, perform one step of steepest descent with exact line search.

(b) Starting at $\boldsymbol{x}^0$, perform one step of Newtons method with exact line search.

(c) What is the global optimal solution? Have any of the methods reached this? What do you think that the next step of the methods would yield? Which method performs best on this problem?