Financial Risk: Credit Risk, Lecture 3

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Content of Lecture

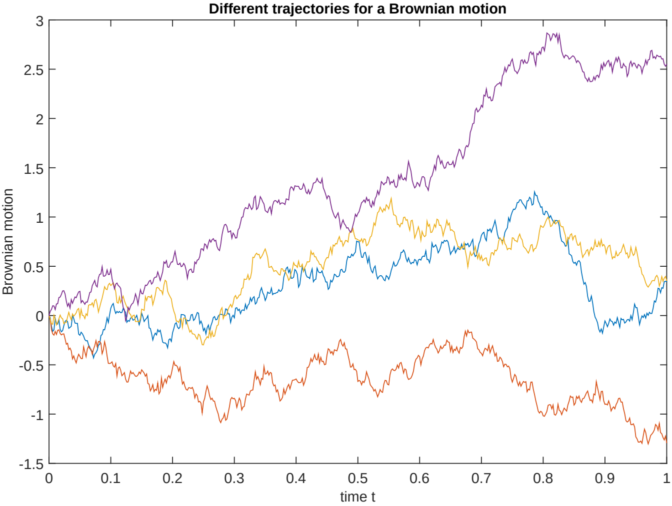
- Discussion of a mixed binomial model inspired by the Merton model
- Derive the large-portfolio approximation formula in this framework
- Discussion how to incorporate random losses in the mixed binomial loss model

Stochastic processes and the Brownian motion

- A continuous-time stochastic process $(Z_t)_{t\in[0,\infty)}$, is a collection of random variables indexed by time $t \in [0,\infty)$,
- For a given random outcome, a continuous-time stochastic process Z_t can be seen as a function of time $t \geq 0$
- Example of a continuous-time stochastic process is the Brownian motion $(W_t)_{t\geq 0}$ sometimes also denoted a Wiener process.
- The following holds for a Brownian motion $(W_t)_{t\geq 0}$
  1. $W_0 = 0$
  2. $(W_t)_{t\geq 0}$ has a continuous path with probability one
  3. For $0 \leq s < t$ then $W_t - W_s \sim N(0, t-s)$, i.e. $W_t - W_s$ is normally distributed with zero mean and variance $t - s$.
  4. $(W_t)_{t\geq 0}$ has independent increments, i.e. for any time points $0 < s_1 < t_1 \leq s_2 < t_2$ then $W_{t_1} - W_{s_1}$ is independent of $W_{t_2} - W_{s_2}$

Brownian motion, cont.

Different trajectories for a Brownian motion W_t



The mixed binomial model inspired by the Merton Model

- Consider a credit portfolio model, not necessary homogeneous, with m obligors, and where each obligor can default up to fixed time point, say T.
- Assume that each obligor i (think of a firm named i) follows the Merton model, i.e. the value of obligor i-s asset $V_{t,i}$ at time t follows the dynamics

$$dV_{t,i} = \mu_i V_{t,i} dt + \sigma_i V_{t,i} dB_{t,i} \qquad (1)$$

where $B_{t,i}$ is a stochastic process defined as

$$B_{t,i} = \sqrt{\rho} W_{t,0} + \sqrt{1-\rho} W_{t,i} \qquad (2)$$

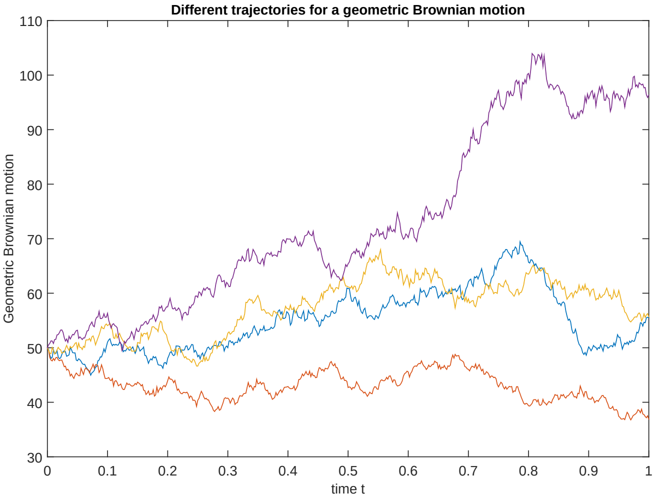
where $\rho \in [0,1]$ and $W_{t,0}, W_{t,1}, \ldots, W_{t,m}$ are independent standard Brownian motions.

- It is then possible to show that $B_{t,i}$ is also a standard Brownian motion. Hence, due to (1) we then know that $V_{t,i}$ is a GBM so by using Ito´s lemma, we get

$$V_{t,i} = V_{0,i} e^{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i B_{t,i}} \qquad (3)$$

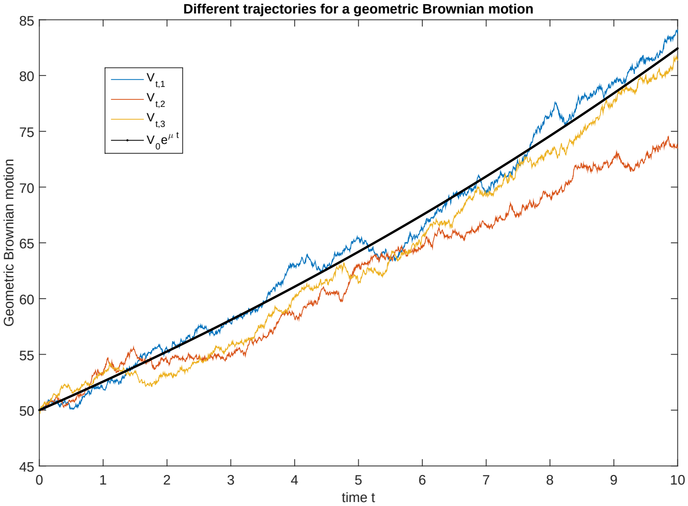
Geometric Brownian motion

Different trajectories for a Geometric Brownian motion $V_t = V_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ for $V_0 = 50, \mu = 0.05, \sigma = 0.25$ (Brownian motion same as on slide 4)



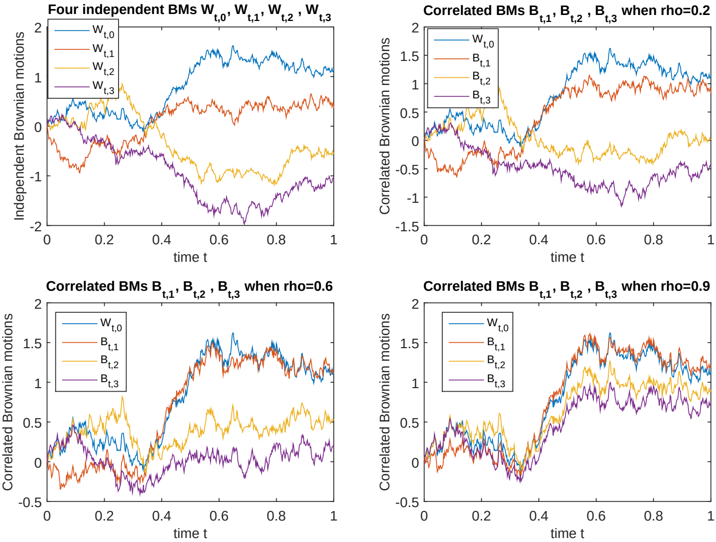
Geometric Brownian motion, cont

Different trajectories for a Geometric Brownian motion $V_t = V_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ for $V_0 = 50, \mu = 0.05, \sigma = 0.025$ and the function $V_0 e^{\mu t}$



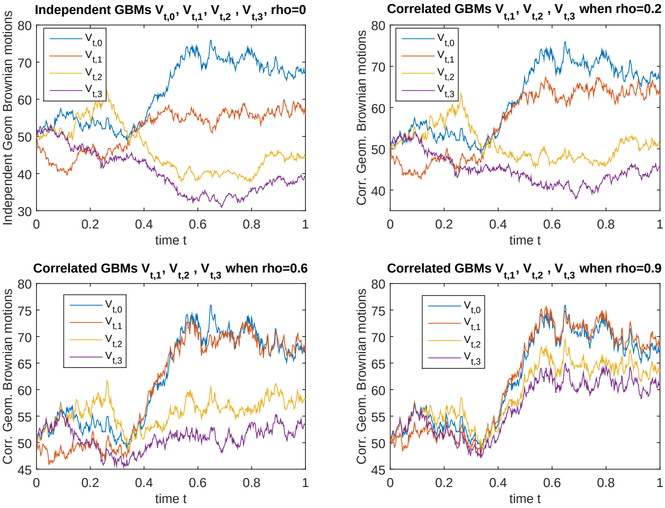
Correlated Brownian motions $B_{t,i}$

Correlated Brownian motions $B_{t,i}$, $i = 1, 2, 3$, given by (2) for different ρ



Correlated Geometrical Brownian motions $V_{t,i}$

Correlated geom. Brownian motions $V_{t,i}$ as in (3) when $B_{t,i}$ as in (2) for different ρ, and same as in prev. slide. $V_{t,i} = 50$, $\mu_i = 0.05$, $\sigma_i = 0.25$ for each $i = 1, 2, 3$



The mixed binomial model inspired by the Merton Model

- The intuition behind (1) and (2) is that the asset for each obligor i is driven by a common process $W_{t,0}$ representing the economic environment, and an individual process $W_{t,i}$ unique for obligor i, where $i = 1, 2, \ldots, m$.
- This means that the asset for each obligor i, depend both on a macroeconomic random process (common for all obligors) and an idiosyncratic random process (i.e. unique for each obligor). This will create a **dependence** among these obligors. To see this, recall that $\text{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j]$ so due to (2)

$$\begin{aligned}
\text{Cov}(B_{t,i}, B_{t,j}) &= \mathbb{E}[B_{t,i}B_{t,j}] - \mathbb{E}[B_{t,i}]\mathbb{E}[B_{t,j}] \\
&= \mathbb{E}\left[\left(\sqrt{\rho}W_{t,0} + \sqrt{1-\rho}W_{t,i}\right)\left(\sqrt{\rho}W_{t,0} + \sqrt{1-\rho}W_{t,j}\right)\right] \\
&= \mathbb{E}\left[\rho W_{t,0}^2\right] + \sqrt{\rho}\sqrt{1-\rho}\left(\mathbb{E}[W_{t,0}W_{t,i}] + \mathbb{E}[W_{t,0}W_{t,j}]\right) \\
&\quad + (1-\rho)\mathbb{E}[W_{t,j}W_{t,i}] \\
&= \rho\mathbb{E}\left[W_{t,0}^2\right] = \rho t
\end{aligned}$$

where the third equality is due to $\mathbb{E}[W_{t,j}W_{t,i}] = 0$ when $i \neq j$.

The mixed binomial model inspired by the Merton Model

- Hence, $\text{Cov}(B_{t,i}, B_{t,j}) = \rho t$ which implies that there is a dependence of the processes that drives the asset values $V_{t,i}$. To be more specific,

$$\text{Corr}(B_{t,i}, B_{t,j}) = \frac{\text{Cov}(B_{t,i}, B_{t,j})}{\sqrt{\text{Var}(B_{t,i})}\sqrt{\text{Var}(B_{t,i})}} = \frac{\rho t}{\sqrt{t}\sqrt{t}} = \rho \qquad (4)$$

  so $\text{Corr}(B_{t,i}, B_{t,j}) = \rho$ which is the mutual dependence among the obligors created by the macroeconomic latent variable $W_{t,0}$

- Note that if $\rho = 0$, we have $\text{Corr}(B_{t,i}, B_{t,j}) = 0$ which makes the asset values $V_{t,1}, V_{t,2}, \ldots, V_{t,m}$ independent (so the obligors are independent).
- Next, let D_i be the debt level for each obligor i and recall from the Merton model that obligor i defaults if $V_{T,i} \leq D_i$, that is if

$$V_{0,i} e^{(\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i B_{T,i}} < D_i \qquad (5)$$

  which, by using the definition of $B_{t,i}$ is equivalent with the event

$$\ln V_{0,i} - \ln D_i + (\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i \left(\sqrt{\rho} W_{T,0} + \sqrt{1-\rho} W_{T,i}\right) < 0 \qquad (6)$$

The mixed binomial model inspired by the Merton Model

- Recall that for each i, $W_{T,i} \sim N(0, T)$, i.e $W_{T,i}$ is normally distributed with zero mean and variance T. Hence, if $Y_i \sim N(0,1)$, $W_{T,i}$ has the same distribution as $\sqrt{T}Y_i$ for $i = 0, 1, \ldots, m$ where $Y_0, Y_1, \ldots, Y_m$ also are independent. Define Z as Y_0, i.e $Z = Y_0$. Then, (6) has same probability as

$$\ln V_{0,i} - \ln D_i + (\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i\left(\sqrt{\rho}\sqrt{T}Z + \sqrt{1-\rho}\sqrt{T}Y_i\right) < 0 \quad (7)$$

and dividing with $\sigma_i\sqrt{T}$ renders

$$\frac{\ln V_{0,i} - \ln D_i + (\mu_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}} + \sqrt{\rho}Z + \sqrt{1-\rho}Y_i < 0. \quad (8)$$

We can rewrite the inequality (8) as

$$Y_i < \frac{-\left(C_i + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}} \quad (9)$$

where C_i is a constant given by

$$C_i = \frac{\ln\left(V_{0,i}/D_i\right) + (\mu_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}} \quad (10)$$

The mixed binomial model inspired by the Merton Model

- Hence, from the previous slides we conclude that

$$V_{T,i} < D_i \quad \text{has same prob/cond.prob as} \quad Y_i < \frac{-\left(C_i + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}} \tag{11}$$

where C_i is a constant given by (10). Next define X_i as

$$X_i = \begin{cases} 1 & \text{if } V_{T,i} < D_i \\ 0 & \text{if } V_{T,i} > D_i \end{cases} \tag{12}$$

- Then (11) implies that

$$\begin{aligned} \mathbb{P}\left[X_i = 1 \,|\, Z\right] = \mathbb{P}\left[V_{T,i} < D_i \,|\, Z\right] &= \mathbb{P}\left[Y_i < \frac{-\left(C_i + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}} \,\middle|\, Z\right] \\ &= N\left(\frac{-\left(C_i + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \end{aligned} \tag{13}$$

where $N(x)$ is the distribution function of a standard normal distribution.

- The last equality in (13) follows from the fact that $Y_i \sim N(0,1)$ and that Y_i is independent of Z in (11).

The mixed binomial model inspired by the Merton Model

- Next, assume that all obligors in the model are identical, so that $V_{0,i} = V_0$, $D_i = D$, $\sigma_i = \sigma$, $\mu_i = \mu$ and thus $C_i = C$ for $i = 1, 2, \ldots, m$.
- Then we have a homogeneous static credit portfolio, where we consider the time period up to T.
- Furthermore, Equation (13) implies that

$$\mathbb{P}[X_i = 1 \mid Z] = N\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \tag{14}$$

  where C is a constant given by (10) with $V_{0,i} = V_0$, $D_i = D$, $\sigma_i = \sigma$, $\mu_i = \mu$ and thus $C_i = C$ for all obligors i.
- Let Z be the "economic background variable" in our homogeneous portfolio and define $p(Z)$ as

$$p(Z) = N\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \tag{15}$$

  where $N(x)$ is the distribution function of a standard normal distribution.

The mixed binomial model inspired by the Merton Model

- Since, $p(Z) \in [0, 1]$, we would like to use $p(Z)$ in a mixed binomial model.

- To be more specific, let $X_1, X_2, \dots X_m$ be identically distributed random variables such that $X_i = 1$ if obligor i defaults before time T and $X_i = 0$ otherwise.

- Furthermore, conditional on Z, the random variables $X_1, X_2, \dots X_m$ are independent and each X_i have default probability $p(Z)$, that is

$$\mathbb{P}\left[X_i = 1 \,|\, Z\right] = p(Z) = N\left(\frac{-\left(C + \sqrt{\rho} Z\right)}{\sqrt{1-\rho}}\right). \tag{16}$$

- We call this the mixed binomial model inspired by the Merton model or sometimes simply a mixed binomial Merton model.

The mixed binomial model inspired by the Merton Model

- Recall that the total credit loss in the portfolio at time T, called L_m, is

$$L_m = \sum_{i=1}^{m} \ell X_i = \ell \sum_{i=1}^{m} X_i = \ell N_m \quad \text{where } N_m = \sum_{i=1}^{m} X_i$$

- In the mixed binomial Merton model Z is a continuous random variable on $\mathbb{R}$ so from last lecture we know that the loss distribution $F_{L_m}(x)$ is given by

$$F_{L_m}(x) = \sum_{k=0}^{\left\lfloor \frac{x}{\ell} \right\rfloor} \int_{-\infty}^{\infty} \binom{m}{k} p(z)^k (1-p(z))^{m-k} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz. \qquad (17)$$

where $p(u) = N\left(\frac{-(C+\sqrt{\rho}u)}{\sqrt{1-\rho}}\right)$

- However, if m is "large" we have the following approximation for the loss distribution $F_{L_m}(x) = \mathbb{P}\left[L_m \le x\right]$

$$F_{L_m}(x) \approx F\left(\frac{x}{\ell m}\right) \quad \text{if } m \text{ is "large".} \qquad (18)$$

for any $x \in [0, \ell m]$ and where $F(x) = \mathbb{P}\left[p(Z) \le x\right]$.

The mixed binomial Merton model and LPA, cont.

- We therefore next want to find an explicit expression of $F(x)$ where $F(x) = \mathbb{P}\left[p(Z) \leq x\right]$. From (16) we know that $p(Z) = N\left(\frac{-(C+\sqrt{\rho}Z)}{\sqrt{1-\rho}}\right)$ where Z is a standard normal random variable, i.e. $Z \sim N(0,1)$.
- Hence, $F(x) = \mathbb{P}\left[p(Z) \leq x\right] = \mathbb{P}\left[N\left(\frac{-(C+\sqrt{\rho}Z)}{\sqrt{1-\rho}}\right) \leq x\right]$ so

$$
\begin{aligned}
\mathbb{P}\left[N \quad \frac{-\left(C+\sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \leq x\Bigg] &= \mathbb{P}\left[\frac{-\left(C+\sqrt{\rho}Z\right)}{\sqrt{1-\rho}} \leq N^{-1}(x)\right] \\
&= \mathbb{P}\left[-Z \leq \frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) + C\right)\right] \\
&= N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) + C\right)\right)
\end{aligned}
$$

where the last equality is due to $\mathbb{P}\left[-Z \leq x\right] = \mathbb{P}\left[Z \geq -x\right] = 1 - \mathbb{P}\left[Z \leq -x\right]$ and $1 - N(-x) = N(x)$ for any x, due to the symmetry of a standard normal random variable.

The mixed binomial Merton model and LPA, cont.

- Hence, $F(x) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) + C\right)\right)$ so what is left is to find C.
- Since our model is inspired by the Merton model, we have that

$$X_i = \begin{cases} 1 & \text{if } V_{T,i} < D \\ 0 & \text{if } V_{T,i} > D \end{cases} \tag{19}$$

so $\mathbb{P}[X_i = 1] = \mathbb{P}[V_{T,i} < D]$. However, from (8) and (11) we conclude that

$$\mathbb{P}[X_i = 1] = \mathbb{P}[V_{T,i} < D] = \mathbb{P}\left[\sqrt{\rho}Z + \sqrt{1-\rho}Y_i \leq -C\right] \tag{20}$$

where C is given by Equation (10) in the homogeneous case where $V_{0,i} = V_0$, $D_i = D$, $\sigma_i = \sigma$, $\mu_i = \mu$ and consequently $C_i = C$ for $i = 1, 2, \ldots, m$.

Furthermore, since Z and Y_i are standard normals then $\sqrt{\rho}Z + \sqrt{1-\rho}Y_i$ will also be standard normal. Hence, $\mathbb{P}\left[\sqrt{\rho}Z + \sqrt{1-\rho}Y_i \leq -C\right] = N(-C)$ and this observation together with (20) implies that

$$\mathbb{P}[X_i = 1] = \mathbb{P}[V_{T,i} < D] = N(-C). \tag{21}$$

The mixed binomial Merton model and LPA, cont.

- Recall that $\bar{p} = \mathbb{E}\left[p(Z)\right] = \int_{-\infty}^{\infty} p(z) f_Z(z) dz$ so $\bar{p} = \mathbb{P}\left[X_i = 1\right]$ since $\mathbb{P}\left[X_i = 1 \,|\, Z\right] = p(Z)$ and thus

$$\mathbb{P}\left[X_i = 1\right] = \mathbb{E}\left[\mathbb{P}\left[X_i = 1 \,|\, Z\right]\right] = \mathbb{E}\left[p(Z)\right] = \bar{p}$$

- Hence, from (21) we have $\bar{p} = N\left(-C\right)$ so

$$C = -N^{-1}\left(\bar{p}\right) \tag{22}$$

  which means that we can ignore C (and thus also ignore V_0, D, σ and μ, see (10)) and instead directly work with the default probability $\bar{p} = \mathbb{P}\left[X_i = 1\right]$. Hence, we estimate $\bar{p}$ to 5%, say, which then implicitly defines the quantizes V_0, D, σ and μ via (10) and (22).

- Finally, going back to $F(x) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) + C\right)\right)$ and using (22) we conclude that

$$F(x) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) - N^{-1}\left(\bar{p}\right)\right)\right) \tag{23}$$

  where $F(x) = \mathbb{P}\left[p(Z) \leq x\right]$.

The mixed binomial Merton model and LPA, cont.

- Hence, if m is large enough, we can in the mixed binomial model inspired by the Merton model, use (32) to get the following approximation for the loss distribution $F_{L_m}(x) = \mathbb{P}[L_m \le x]$

$$\mathbb{P}[L_m \le x] \approx N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}\left(\frac{x}{\ell m}\right) - N^{-1}(\bar{p})\right)\right) \qquad (24)$$

  where $\bar{p} = \mathbb{P}[X_i = 1]$ is the individual default probability for each obligor.

- The approximation (23) or equivalently, (24) is sometimes denoted the **LPA** in a static Merton framework, and was first introduced by Vasicek 1991, at KMV, in the paper *"Limiting loan loss probability distribution"*.

- The **LPA** in a Merton framework and its offsprings (i.e. variants) is today **widely** used in the industry (Moody's-KMV, CreditMetrics etc. etc.) for risk management of large credit/loan portfolios, especially for computing regulatory capital in **Basel II** and **Basel III** (Basel III is currently being implemented (since end of 2013)).

The mixed binomial Merton model: The role of ρ

- Recall from (4) that ρ was the correlation parameter describing the dependence between the Brownian motions $B_{t,i}$ that drives each obligor i's asset price, i.e. $\text{Cov}(B_{t,i}, B_{t,j}) = \rho t$ for all $t > 0$ so $\text{Corr}(B_{t,i}, B_{t,j}) = \rho$.

- Recall that $X_1, X_2, \ldots, X_m$ was defined as in (19), that is

$$X_i = \begin{cases} 1 & \text{if } V_{T,i} < D \\ 0 & \text{if } V_{T,i} > D \end{cases}$$

  where $V_{T,i}$ is the asset given by (3) in the homogeneous case.

- One can show that for $i \neq j$ then (see in the lecture notes for details)

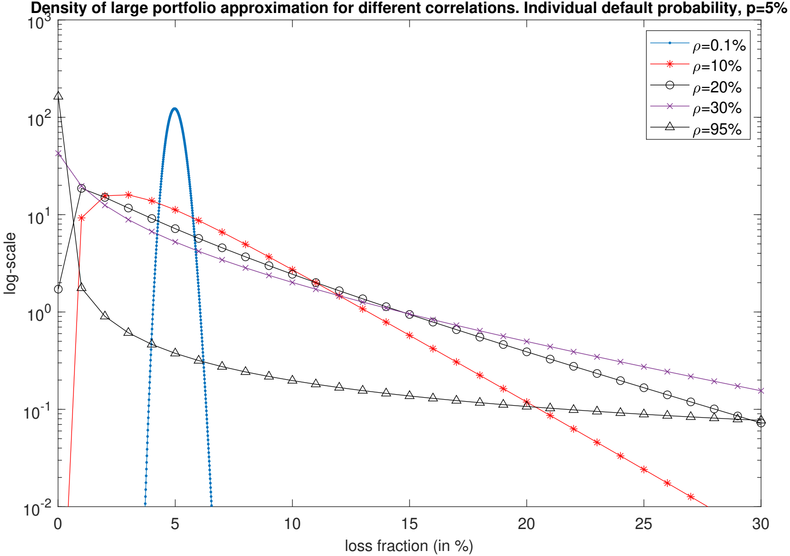
$$\text{Cov}(X_i, X_j) = 0 \quad \text{if } \rho = 0 \tag{25}$$

  and

$$\text{Cov}(X_i, X_j) > 0 \quad \text{if } \rho > 0. \tag{26}$$

- We therefore conclude that ρ is a measure of **default dependence** among the zero-one variables $X_1, X_2, \ldots, X_m$ in the mixed binomial Merton model.

The mixed Merton binomial model and LPA, cont.



The mixed Merton binomial model and LPA, cont.

- Given the limiting distribution $F(x)$

$$F(x) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) - N^{-1}(\bar{p})\right)\right) \tag{27}$$

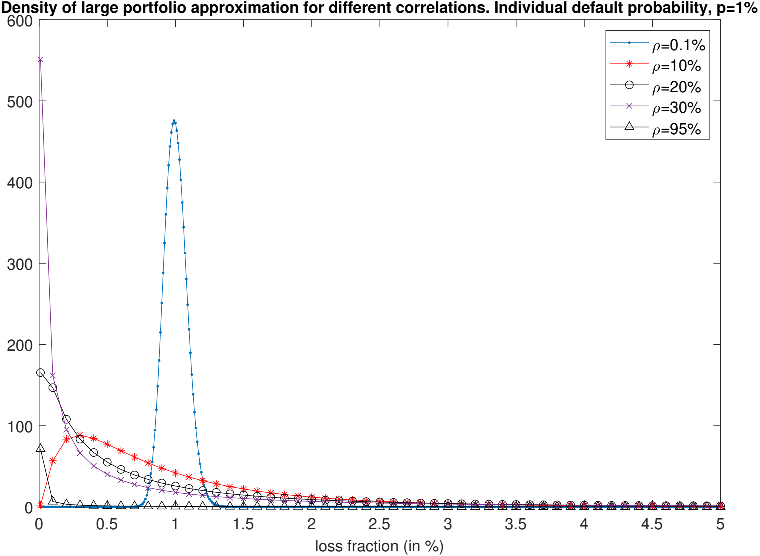
we can also find the density $f_{\text{LPA}}(x)$ of $F(x)$, that is $f_{\text{LPA}}(x) = \frac{dF(x)}{dx}$.

- It is possible to show that

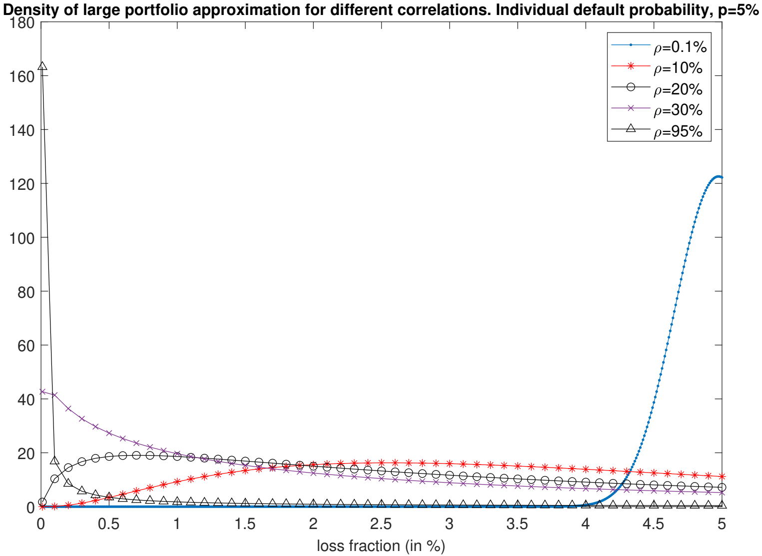
$$f_{\text{LPA}}(x) = \sqrt{\frac{1-\rho}{\rho}}\exp\left(\frac{1}{2}(N^{-1}(x))^2 - \frac{1}{2\rho}\left(N^{-1}(\bar{p}) - \sqrt{1-\rho}N^{-1}(x)\right)^2\right) \tag{28}$$

- This density is just an approximation, and fails for small number of the loss fraction.

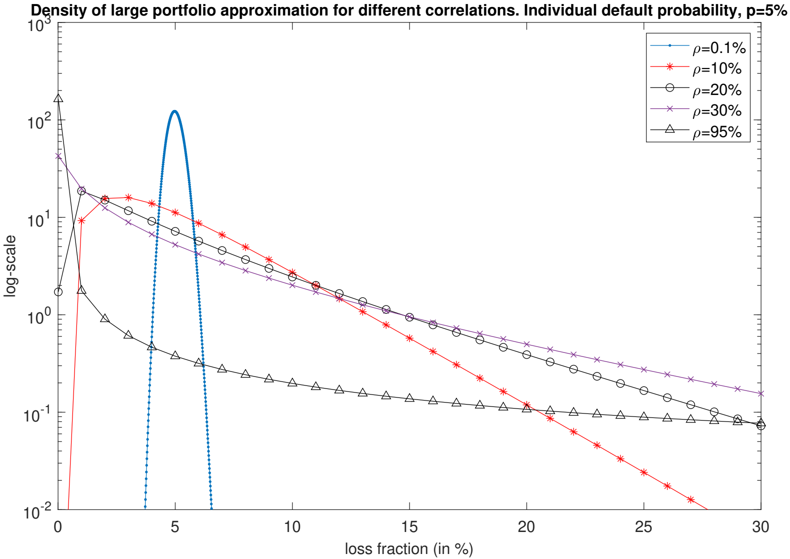
The mixed Merton binomial model and LPA, cont.



The mixed Merton binomial model and LPA, cont.



The mixed Merton binomial model and LPA, cont.



VaR in the mixed binomial Merton model

Consider a static credit portfolio with m obligors in a mixed binomial model inspired by the Merton framework where

- the individual one-year default probability is $\bar{p}$
- the individual loss is ℓ
- the default correlation is ρ

By assuming the LPA setting we can now state the following result for the one-year credit Value-at-Risk $\text{VaR}_\alpha(L)$ with confidence level $1-\alpha$.

VaR in the mixed binomial Merton model using the LPA setting

With notation and assumptions as above, the one-year $\text{VaR}_\alpha(L)$ is given by

$$\text{VaR}_\alpha(L) = \ell \cdot m \cdot N\left(\frac{\sqrt{\rho}N^{-1}(\alpha) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right). \tag{29}$$

Useful exercise: Derive the formula (29).

Note that variants of the formula (29) is extensively used for computing regulatory capital in **Basel II** and **Basel III**

Random losses in the mixed binomial loss model

- In the last three lectures the individual loss ℓ_i for each obligor i have been a constant ℓ same for all obligors, when studying the mixed binomial loss model, that is $\ell = \ell_1 = \ell_2 = \ldots = \ell_m$
- It is possible to extend the mixed binomial loss models to allow for random losses ℓ_i for each obligor $i = 1, 2, \ldots, m$
- By homogeneity, the distribution of these losses must be same for all obligors, and by linearity of VaR, the losses are in percent, i.e. values in $[0, 1]$
- Let Z be the mixing distribution in a mixed binomial model with individual default probability $p(Z) = \mathbb{P}[X_i = 1 \,|\, Z]$ same for all obligors.
- One way to introduce random losses, is to let the individual losses $\ell_i(Z)$ be random variables which conditional on Z, are i.i.d, all having the distribution $\ell(Z)$ for some function $\ell(x) \in [0, 1]$ for all x
- Hence, conditionally on Z, then $\ell_1(Z), \ell_2(Z), \ldots, \ell_m(Z)$ are i.i.d with distribution given by $\ell(Z)$

Random losses in the mixed binomial loss model, cont.

- The portfolio loss L_m will now be given by $L_m = \sum_{i=1}^{m} \ell_i(Z)X_i$
- Depending on the nature of the individual loss distribution $\ell(Z)$ one can sometimes get closed form expressions for the exact loss distribution $F_{L_m}(x) = \mathbb{P}[L_m \leq x]$, for example if $\ell(Z)$ is a discrete distribution
- Conditionally on Z, the random variables $\ell_1(Z)X_1, \ell_2(Z)X_2, \ldots \ell_m(Z)X_m$ are i.i.d with distribution $\ell(Z)p(Z)$.
- Thus, conditionally on Z we can use the law of large numbers for $\frac{L_m}{m}$ to conclude that

$$\text{given a "fixed" outcome of } Z \text{ then } \frac{L_m}{m} \to \ell(Z)p(Z) \text{ as } m \to \infty \quad (30)$$

- Since a.s convergence implies convergence in distribution then (30) implies that for any $x \in [0,1]$ we have

$$\mathbb{P}\left[\frac{L_m}{m} \leq x\right] \to \mathbb{P}[\ell(Z)p(Z) \leq x] \quad \text{when} \quad m \to \infty. \quad (31)$$

Random losses in the mixed binomial loss model, cont.

- We also have for any $x \in [0, \infty)$, or in fact any $x \in [0, m]$ (why?) that

$$F_{L_m}(x) = \mathbb{P}[L_m \leq x] = \mathbb{P}\left[\frac{L_m}{m} \leq \frac{x}{m}\right]$$

and this in (31) then implies that

$$F_{L_m}(x) \to \mathbb{P}\left[\ell(Z)p(Z) \leq \frac{x}{m}\right] \quad \text{as } m \to \infty$$

where we recall that $\ell(Z) \in [0, 1]$.

- Hence, if m is "large" we have the following approximation

$$F_{L_m}(x) \approx \mathbb{P}\left[\ell(Z)p(Z) \leq \frac{x}{m}\right] \qquad \text{for any } x \in [0, m] \tag{32}$$

- Depending on the nature of $\ell(Z)$ one can sometimes get closed form expressions of $\mathbb{P}[\ell(Z)p(Z) \leq x]$, for example if $\ell(Z)$ is a discrete distribution
- However, we can always find an estimation of L_m by simulating the random variables Z and $X_1, \ldots, X_m$ and $\ell_1(Z), \ell_2(Z), \ldots \ell_m(Z)$

Monte-Carlo simulations

Monte-Carlo simulation of the portfolio credit loss

Let n be the number of simulations

For each $j = 1, 2, \dots, n$, repeat the following five steps:

1. Simulate the random variable Z and compute $p(Z) \in [0, 1]$.
2. Simulate the i.i.d sequence $U_1, U_2, \dots, U_m$ where U_i is uniformly distributed on $[0, 1]$ and independent of Z.
3. For each $i = 1, 2, \dots, m$ define X_i as

$$X_i = \begin{cases} 1 & \text{if } U_i \leq p(Z) \\ 0 & \text{otherwise, i.e. if } U_i > p(Z) \end{cases} \tag{33}$$

4. If losses are random, simulate $\ell_1(Z), \ell_2(Z), \dots \ell_m(Z)$
5. Compute $L_m^{(j)} = \sum_{i=1}^m X_i \ell_i(Z)$.

From the simulated sequence $\{L_m^{(j)}\}_{j=1}^n$ we can find the empirical distribution function and use it to find an estimate of Value-at-Risk etc.

Monte-Carlo simulations, cont.

Let us motivate why (33) for generating $X_1, \ldots, X_m$ implies that $\mathbb{P}[X_i = 1 \mid Z] = p(Z)$ for each $i = 1, 2, \ldots, m$.

Let $F_{U_i}(x) = x$ be the distribution function for U_i which is uniformly distributed on $[0, 1]$.

Given $p(Z)$ we then have by construction that

$$\mathbb{P}[X_i = 1 \mid Z] = \mathbb{P}[U_i \leq p(Z) \mid Z] = F_{U_i}(p(Z)) = p(Z) \tag{34}$$

where second equality is due to fact that U_i is independent of Z. The final equality in (34) follows from $F_{U_i}(x) = x$ since U_i is uniformly distributed on $[0, 1]$.

This proves (33).