May 10, 2021

Lecture notes on non-linear ODEs: existence, extension, limit sets, periodic solutions.

Plan

1. Peano theorem on existence of solutions (without proof), Theorem. 4.2, p. 102.

2. Existence and uniquness theorem by Picard and Lindelöf . Th. 4.17, p. 118 (for continuous $f(t,x)$, locally Lipschitz in x), (Proof comes in the last week of the course)

Th.4.22, p.122 (for piecewise continuous $f(t,x)$, locally Lipschitz in x).

3. Maximal solutions. Openness of the maximal existence interval. Prop. 4.4., p. 107.

4. Existence of Maximal solutions. Theorem 4.8.

5. Extensibility of bounded solutions to the boundary time point of the interval. Lemma 4.9, p. 110.

6. Corollary 4.10, p. 111, on solutions enclosed in a compact, implying "infinite" maximal interval.

7. Properties of limits of maximal solutions. Theorem 4.11, p. 112 on the property of solutions with "finite" maximal interval $I_{\max}$, to escape any compact subset C in the space domain $C \subset G$.

8. On infinite existence interval for systems with linear growth estimate for the right hand side. Proposition 4.12, p. 114.

9. Transition map. Definition p. 126. Transition property of the transition map. Translation property for autonomous systems.

Theorem 4.26, p. 126. (similar to Chapman - Kolmogorov relations for transition matrix)

10. Openness of the domain and smoothness of transition map.Theorem 4.29, p. 129.

11. Autonomous systems. §4.6.1. Example 4.33., p. 139. of a transition map.

Lecture 14
Plan

1. Repetition of existence results.Uniqueness.
2. Extension of solutions to a maximal interval. Exercises on maximal intervals.
3. Lemma on extention of solution to a boundary point of the time interval.
4. Corollary about "eternal life" for solutions contained in a compact.
5. Corollary about escaping of compacts for "short living" solutions.

0.1 Non-linear systems. Existence and uniqueness of solutions.

Second half of the course deals with initial value problems for non-linear systems of ODE's, non-autonomous:

$$x'(t) = f(t, x), \quad f : J \times G \to \mathbb{R}^n; \quad x(\tau) = \xi \tag{1}$$

with $J \subset \mathbb{R}$ - an interval, $G \subset \mathbb{R}^n$, open, $\tau \in J$, $\xi \in G$, f- continuous in $J \times G$,

The set $J \times G$ is a "cylinder" in $\mathbb{R}^{n+1}$ with the bottom G. In $\mathbb{R}^2$ if G is an interval $J \times G$ is just a rectangle.

and autonomous systems of ODE's:

$$x'(t) = f(x), \quad f : G \to \mathbb{R}^n; \quad x(\tau) = \xi \tag{2}$$

that are a particular case of (1) with $G \subset \mathbb{R}^n$, open, $\tau \in J = \mathbb{R}$, $\xi \in G$, f- continuous in G, where the right hand side f in the equation is independent of the time variable t running over the whole $\mathbb{R}$. The practical meaning of this kind of systems is that the "velocity" f of the system depends only on the position x, but not on time t. So independently of the starting time τ the output $x(t)$ of an evolution depends only on the shift in time $t - \tau$. It lets to choose always $\tau = 0$ for autonomous systems.

In many situations the equivalent integral form of I.V.P. is convenient to use:

$$x(t) = \xi + \int_{\tau}^{t} f(s, x(s))ds \tag{3}$$

Another option for requirements to f that is considered in the book by Logemann Ryan is that f is supposed to be piecewise continuous in t and locally Lipschitz with respect to x. We will not consider this case systematically in this part of the course.

Fixed point problems

The existence of solutions to abstract non - linear equations in the form of a so called **fixed point problem**

$$z = \mathcal{B}\{z\}$$

for an operator $\mathcal{B} : H \to H$ defined on a complete vector space H (Banach space) is resolved by one of two general methods.

1. **Compactness principles.** One of examples of this scope of ideas is the theorem by Schauder. For a continuous operator $\mathcal{B} : K \to K$ defined on a convex closed subset K of H and mapping it to a compact set $\mathcal{B}\{K\}$, there is at least one fixed point z in K that is a solution to the equation $z = \mathcal{B}\{z\}$.

2. **Banach's contraction principle.** Convergence of successive approximations: $z_{n+1} = \mathcal{B}\{z_n\}$ to a fixed point for a "small" operator $\mathcal{B}$.

The fundamental question of existence of solutions is answered by the following Peano theorem (with possibility of non-uniqueness of solutions)

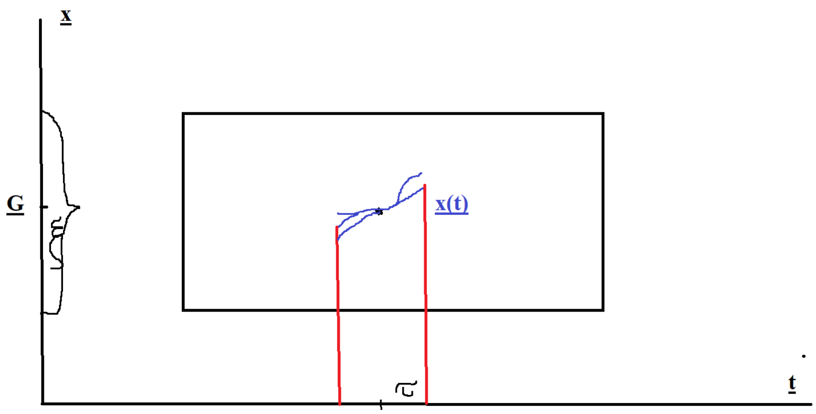
Theorem 4.2, p. 102. Peano theorem.

For each (τ, ξ) in $J \times G$ there exists at least one solution to (1) defined on a (possibly small) time interval $I \subset J$, $\tau \in I$.

Point out that solutions are not unique leading to branching of the trajectory of solution on the picture.

This result implies also the solvability of the problem (2) that is just a particular case.

The proof of this theorem is based on the compactness principle, one of two main approaches in analysis to the existence of solutions to non-linear equations. We do not give a



proof, but will sketch main ideas behind it.

i) One of characteristic properties of compact sets in complete normed spaces is, that any sequence of points $\{z_n\}_{n=1}^{\infty}$ from a compact set C always has a converging subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ with a limit $\lim_{k\to\infty} z_{n_k} = z_*$ that belongs to C: $z_* \in C$.

ii) One approximates solutions to (1) by the explicit Euler method;

$$x(t) = x_k + (t - t_k) f(t_k, x(t_k)), \quad t \in (t_k, t_{k+1})$$

and considers a sequence $\{y_n(t)\}_{n=1}^{\infty}$ of such approximations if steps $(t_{k+1} - t_k)$ in finite differences tend uniformly to zero with $n \to \infty$. Such an approximation in one dimensional case has a graph in the form of continuous peacewise linear broken line.

iii) Considering these approximations on a time interval I including τ and choosing this interval small enough (depending on the absolute value of f around (τ, ξ)), one can show that the approximathions $\{y_n(t)\}_{n=1}^{\infty}$, are uniformly bounded and uniformly continuous on I.

iv) Then basing on the property i) and on iii), one can choose a subsequence $\{y_{n_k}(t)\}_{k=1}^{\infty}$ converging uniformly on I, in the space of continuous vector valued functions on I , to a continuous function $y(t)$ that is a solution to (3) and therefore to (1).$\square$

Remark. Choosing different converging subsequences in this construction can in general lead to different limits and to non-unique solutions that is seen as branching of the trajectory of solution on the picture.

Exercise. Show that the I.V.P. $x' = \sqrt[3]{x}$; $x(0) = 0$, has non-unique solutions.

The uniqueness of solutions to I.V.P. needs additional requirements on regularity of $f(t, x)$ with respect to x variable. A standard requirement is that $f(t, x)$ is supposed to be locally Lipschitz with respect to the space x variable.

We repeat here the definition of locally Lipschitz functions.

Definition of a relatively open set.

Let A be any subset in a metric space X. The the set U_A is called to be relatively open in A if there is an open subset $U \subset X$ such that $U_A = U \cap A$.

Definition.(p. 115) Locally Lipschitz function

Let $D \subset \mathbb{R}^n$ be a non-empty set. A function $g : D \to \mathbb{R}^M$ is said to be locally Lipschitz if for any $z \in D$ there is a set $U \subset D$, relatively open in D, $z \in U$, and a number $L \geq 0$ (which may depend on U) such that

$$\|g(u) - g(w)\| \leq L \|u - w\|, \quad \forall u, w \in U$$

If L is independent of the choice of U, the function is called globally Lipschitz.

Similarly one defines functions locally Lipschitz with respect to a part of variables.

Definition.(p. 118)

Let $G \subset \mathbb{R}^n$ be a non-empty open set, J be an interval in $\mathbb{R}$. A function $f : J \times G \to \mathbb{R}^n$ is said to be locally Lipschitz with respect to $x \in G$ if for any $(\tau, x) \in J \times G$ there is a set $S \times U \subset J \times G$, relatively open in $J \times G$ and a number $L \geq 0$ (which may depend on $S \times U$) such that

$$\|g(s, x) - g(s, y)\| \leq L \|x - y\|, \quad \forall (s, x), (s, y) \in S \times U$$

A theorem that gives conditions for both existence and uniqness of solutions to (1) is called the Picard-Lindelöf theorem

We will prove it in the last week of the course by applying the Banach contraction principle, that is the second main approach in analysis to existence of solutions to non-linear equations.

Theorem. **Picard-Lindelöf. Theorem 4.17, p. 118 (variant with continuous** f).

Let with $J \subset \mathbb{R}$ - an interval, $G \subset \mathbb{R}^n$, open, $\tau \in J$, $\xi \in G$, f be continuous in $J \times G$. If f is locally Lipschitz with respect to its second argument $x \in G$, then there is a unique **maximal solution** $x : I_x \to \mathbb{R}^n$ to the I.V.P. problem (1). Any other maximal solution with the same initial conditions must coinside with $x(t)$.

Definition. By maximal solution we mean here the solution that cannot be extended to a larger time interval.

Definition. p. 106.

An extension (proper extension) of the solution $x : I \to \mathbb{R}^n$ is a solution $\widetilde{x} : \widetilde{I} \to \mathbb{R}^n$ to the differetial equation (1) such that $\widetilde{x}(t) = x(t)$ $\forall t \in I$, $I \subset \widetilde{I}$, $\widetilde{I} \neq I$.

A simpler version of this theorem states just that a "local" solution to (1) on a possibly small time interval $I \subset J$, $\tau \in I$, exists and is unique in the sense that any two solutions x and y must coinside on the intersection of the time intervals I_x and I_y where they are defined.

Proof of local uniqueness uses the integral form of the problem and the argument with the Grönvall inequality that was in a similar fashion applied two times earlier to linear systems.

The same argument with the Grönvall inequality is used for proving well posedness of the I.V.P., namely that solutions to initial value problem (1) considered as functions of three variables t, τ, ξ: $x(t) = \varphi(t, \tau, \xi)$ are continuous and in fact even locally Lipschitz with respect to all three variables t, τ, ξ.

The uniqueness proof.

Consider difference of two solutions $x(t)$ and $y(t)$ to I.V.P. defined on a set $S \times U$ including (τ, ξ)such that the local Lipschitz property is valid for $f(t, x)$ on $S \times U$.

$$\begin{aligned} x(t) &= \xi + \int_\tau^t f(s, x(s))ds \\ y(t) &= \xi + \int_\tau^t f(s, y(s))ds \end{aligned}$$

$$
\begin{aligned}
x(t) - y(t) &= 0 + \int_\tau^t f(s, x(s)) - f(s, y(s)) ds \\
\|x(t) - y(t)\| &= \left\| \int_\tau^t f(s, x(s)) - f(s, y(s)) ds \right\| \leq \\
&\leq \int_\tau^t \|f(s, x(s)) - f(s, y(s))\| \, ds \leq \int_\tau^t L \, \|x(s) - y(s)\| \, ds \\
&= 0 + L \int_\tau^t \|x(s) - y(s)\| \, ds
\end{aligned}
$$

The Grönvall inequality implies that solutions $x(t)$ and $y(t)$ must coinside:

$$\|x(t) - y(t)\| \leq 0 \cdot \exp(L(t - \tau)) = 0$$

■

0.2 Extensions, maximal solutions and their properties.

The condition in the Proposition 4.12 is not necessary, but simple examples show solutions that blow up in finite time in future or in the past if this condition is not satisfied, as for example the equation $x' = x^2$.

We consider in this section the problem (1) with f continuous and satisfying conditions in the Peano theorem implying existence (but not uniqueness) of "local solutions $x : I \to \mathbb{R}^n$ on an interval $I \subset J$.

Definition. p. 106. Maximal solution and maximal interval of existence.

The interval I is a maximal interval of existence and x is called maximal solution if x does not have an extension to a larger interval that is a solution to the same differential equation (1).

We suggest some simple examples of maximal solutions and maximal intervals that can be calculated explicitely.

Exercise 4.6

$J = [-1, 1]$; $G = R$; $\quad f : J \times G \to R$,

$(\tau, \xi) = (0, 1)$

$$\begin{aligned} z'(t) &= f(t, z) \\ f(t, z) &= \frac{3z^2\sqrt{1-|t|}}{2} \\ J &= [-1, 1] \end{aligned}$$

$t \in [0, 1]$

$$\begin{aligned} \frac{dz}{dt} &= \frac{3z^2\sqrt{1-t}}{2} \\ \frac{dz}{z^2} &= \frac{3\sqrt{1-t}}{2}dt \\ \frac{-1}{z} &= -(1-t)^{3/2} + C \\ -1 &= -1 + C; \quad (\tau, \xi) = (0, 1) \\ C &= 0 \\ z &= \frac{1}{(1-t)^{3/2}}; \quad t \in [0, 1) \end{aligned}$$

The solution cannot be extended to the right "last" time point $t = 1$ in the domain of the equation. It blows up with $t \to \infty$.

$t \in [-1, 0]$;

$$\frac{dz}{dt} = \frac{3z^2\sqrt{1+t}}{2}; \quad t \leq 0$$

$$
\begin{aligned}
\frac{dz}{z^2} &= \frac{3\sqrt{1+t}}{2}dt \\
\frac{-1}{z} &= (1+t)^{3/2} + C \\
-1 &= 1 + C; \quad (\tau, \xi) = (0,1) \\
C &= -2 \\
\frac{-1}{z} &= (1+t)^{3/2} - 2 \\
z &= \frac{1}{2-(1+t)^{3/2}}; \quad t \in [-1, 0];
\end{aligned}
$$

The maximal interval $I_{\max} = [-1, 1)$ - is relatively open in $[-1, 1]$ because

$$[-1, 1) = [-1, 1] \cap (-2, 1)$$

with an open interval $(-2, 1)$.

Exercise 4.7

$J = (-\infty, 1)$; $G = (-\infty, 1).t = 0,\ z = 0$

$$f(t,z) = \frac{1}{\sqrt{(1-t)(1-z)}}$$

$$\begin{aligned}
\frac{dz}{dt} &= \frac{1}{\sqrt{(1-t)(1-z)}} \\
\int \sqrt{1-z}dz &= \int \frac{dt}{\sqrt{(1-t)}} \\
\frac{2}{3}(z-1)\left(\sqrt{1-z}\right) &= -2\sqrt{1-t}+C \\
\frac{2}{3}(-1)(1) &= -2+C; \quad t=0,\ z=0 \\
4/3 &= 2-2/3=C \\
\frac{2}{3}(z-1)\left(\sqrt{1-z}\right) &= -2\sqrt{1-t}+\frac{4}{3} \\
\frac{2}{3}(1-z)\left(\sqrt{1-z}\right) &= 2\sqrt{1-t}-\frac{4}{3} \\
(1-z)\left(\sqrt{1-z}\right) &= 3\sqrt{1-t}-2 \\
(1-z)^{3/2} &= 3\sqrt{1-t}-2 \\
(1-z) &= \left(3\sqrt{1-t}-2\right)^{3/2} \\
z &= 1-\left(3\sqrt{1-t}-2\right)^{3/2}
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to 5/9} x(t) &= 1 \\
I_{\max} &= (-\infty, 5/9)
\end{aligned}$$

$I_{\max}$ is open.

Proposition 4.4. Openness of maximal intervals.

Let $x : I \to G$ be a maximal solution to I.V.P. (1).The maximal interval I is relatively open in J (just open if $J = \mathbb{R}$).

It means that $I = J \cap O$ for some open set $O \subset \mathbb{R}$.

□

Example

For example the interval $[-1, 0.5)$ is relatively open in $[-1, 1)$ and in $[-1, 1]$,because

$$\begin{aligned}(-2, 0.5) \cap [-1, 1) &= [-1, 0.5) \\ (-2, 0.5) \cap [-1, 1] &= [-1, 0.5)\end{aligned}$$

Proof. Consider the case when J is an open interval $J = (a, b)$. Suppose that the maximal interval of a maximal solution to I.V.P. $I \subset J$ is not open, for example is $(\alpha, \omega]$.

In this case the point $(\omega, x(\omega)) \in J \times G$ and there is a solution to the differential equation with initial conditions $(\omega, x(\omega))$, existing on a small time interval $[\omega, \omega + \varepsilon)$ with $\omega + \varepsilon < b$. This solution is an extension of the original solution. It is a contradiction because we supposed that $(\alpha, \omega]$ was a maximal interval for the maximal solution $x(t)$. Other cases are considered similarly.■

Theorem 4.8. p. 108. Existence of maximal solutions.

Every solution to a differential equation (1) can be extended to a maximal solution.

□

Idea of the proof(not required at exam)

In the case when solutions are unique (for example f is locally Lipschitz with respect to x), one can build the maximal interval of existence just as a union of domains for all extensions of a given solution. Because of the uniqueness of solutions, trajectories cannot make branches in this case and this construction leads to a unique maximal solution that at each time point t attains the value of one of the extensions defined at this time point. The uniqueness of solutions makes that this definition is consistent.

In the general case when trajectories can create branches, the union of extensions can have a tree like geometry, or even be an n-dimensional set. In this case the proof uses Zorn lemma (see appendix in the course book) to choose a maximal solution. It has an existence interval including all existence intervals of all extensions, but is possibly not unique.

The following technical lemma is the main tool in several arguments about maximal solutions.

Lemma 4.9. p. 110. On the extension to the boundary point of the open existence time interval for a bounded solution having the closure of the orbit in G,

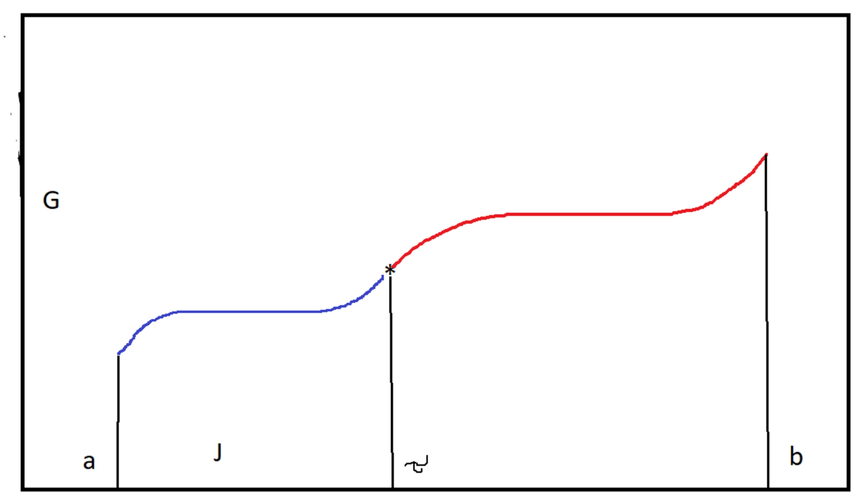
Let $x : I \to G$ be a solution to the differential equation (1) and denote $a = \inf I$; $b = \sup I$.

(1) If b is in J and not in I (I is open in the right end), and the closure $\overline{O_+}$ of the orbit $O_+ = \{x(t) : t \in [\tau, b)\}$ is a **compact subset** of G,(the closure does ot reach the boundary of G)

then there is a solution $y : I \cup \{b\} \to G$ to (1) that is an extension of x.

(2) a similar statement is valid for the "backward orbit" $O_- = \{x(t) : t \in (a, \tau]\}$ and extension of x to the left end point a.

Comment. Compact sets are sets that are bounded and closed.



Proof. We prove (1).

$$x(t) = \xi + \int_{\tau}^{t} f(s, x(s))ds$$

Let C be the closure of the forward orbit $O_+ = \{x(t), t \in [\tau, b)\}$. Assume that $b \in J \setminus I$ and that C is a compact in G.

The continuous function $f(t,z)$ must be bounded on the compact $[\tau, b] \times C$.

$$\|f(t,z)\| < M, \quad (t,z) \in [\tau, b] \times C$$

It will imply that the limit

$$\eta = \lim_{t \to b} \int_\tau^t f(s, x(s)) ds$$

is well defined for continuous and uniformly bounded function under the integral.

We prove it by observing that for any sequence $\{t_k\}_{k=1}^\infty$, $t_k < b$, $t_k \to b$, with $k \to \infty$,integrals $\left\{\int_\tau^{t_k} f(s,x(s))ds\right\}_{k=1}^\infty$ form a Cauchy sequence:

$$\left\|\int_\tau^{t_p} f(s,x(s))ds - \int_\tau^{t_m} f(s,x(s))ds\right\| = \left\|\int_{t_m}^{t_p} f(s,x(s))ds\right\| \leq M\,|t_p - t_m| \to 0, \quad p, m \to \infty$$

that has a limit η independent of the sequence $\{t_k\}_{k=1}^\infty$.

$$\lim_{t \to b} x(t) = \xi + \lim_{t \to b} \int_\tau^t f(s,x(s))ds = \xi + \eta$$

Then the solution $x(t) = \xi + \int_\tau^t f(s,x(s))ds$ to the I.V.P. can be extended to the closed interval $[\tau, b]$ by setting $x(b) = \xi + \eta$.$\blacksquare$

The following Corollary is a direct consequence of the Lemma 4.9 and Proposition 4.4 and gives a sufficient condition for a maximal solution to have an infinite maximal interval (if J is infinite) or a maximal interval "ifinite with respect to" J , which meaning is specified exactly below.

Corollary 4.10, p. 111. "Eternal life" of solutions enclosed in a compact.

Let $x : I_{\max} \to G$ be a maximal solution to (1).

Suppose that the "future" half - orbit $O_+ = \{x(t) : t \in I_{\max} \cap [\tau, \infty)\}$ of the maximal solution $x(t)$ is contained in a compact subset C of G.

Then the corresponding maximal interval of existence $I_{\max}$ is infinite to the right (future) if $[\tau, \infty) \subset J$), or "infinite to the right with respect to J" meaning that the maximal solution exists on $[\tau, \infty) \cap I = [\tau, \infty) \cap J$ that is the whole part of J to the right of the initial time τ.

$\square$

Similar statement is valid for the "backward orbit" $O_{-} = \{x(t) : t \in (a, \tau]\}$. Suppose that the "backward orbit" is contained in a compact subset C of G,

Then the corresponding maximal interval of existence $I_{\max}$ is infinite to the left (past) if $(-\infty, \tau] \subset J$ and is infinite to the left (past) "with respect to" J, that means that the maximal solution exists on $(-\infty, \tau] \cap I = (-\infty, \tau] \cap J$, that is the whole part of J to the left of the initial time τ.

If the whole orbit $O = \{x(t) : t \in I_{\max}\}$ of the maximal solution $x(t)$ is contained in a compact subset of G, then the corresponding maximal interval of existence $I_{\max} = J$ ($I_{\max} = \mathbb{R}$ if $J = \mathbb{R}$). It means that the maximal solution x exists both in the whole past and whole future for the equation.$\square$

Proof. The proof is easy to carry out by a contradiction argument that follows from the Lemma 4.9 and the fact that a maximal interval must be open (relatively to J).

Suppose that the statement of the Corollary is wrong and for example the right end b of the maximal interval $I_{\max}$ is smaller then $\sup J$ (right point in the time domain J). Then the fact that the orbit O_{+} is contained in a compact $C \subset G$ implies that the closure $\overline{O_{+}}$ of O_{+} is compact. It implies by Lemma 4.9, that the solution can be extended to the point b. It is a contradiction, because the maximal interval must be open. Or we just can extend the solution from $t = b$ further to a larger interal $[b, b + \varepsilon)$ by applying the existence theorem for the initial point $(b, x(b))$. It is also a contradiction.

$\blacksquare$

Lecture 15
Summary of Lecture 14

1. Existence theorems. Proof of uniqueness of solutions in the case of locally Lipschitz right hand side in the differential equation.

2. Maximal solutions. Examples.

3. Extension of a trajectory with compact closure of the orbit in the domain - to the boundary point of the open existence interval.

4. "Eternal" existence time for solution with an orbit contained in a compact.

5. How to show that an orbit is contained in a compact without solving the equation. Method with test functions.

How to show that a solution has the orbit inside a compact set?

Definition. A set Q is called positively invariant for a system of differential equations if all trajectories of maximal solutions starting inside Q stay inside Q for all future t in it's maximal interval.

□

We consider here an idea how to show that solutions to a *non-linear autonomous system* of differential equations belong to a compact set.

A general idea that is used to answer many questions about behaviour of solutions (trajectories) of the equations, is the idea of *test functions.*

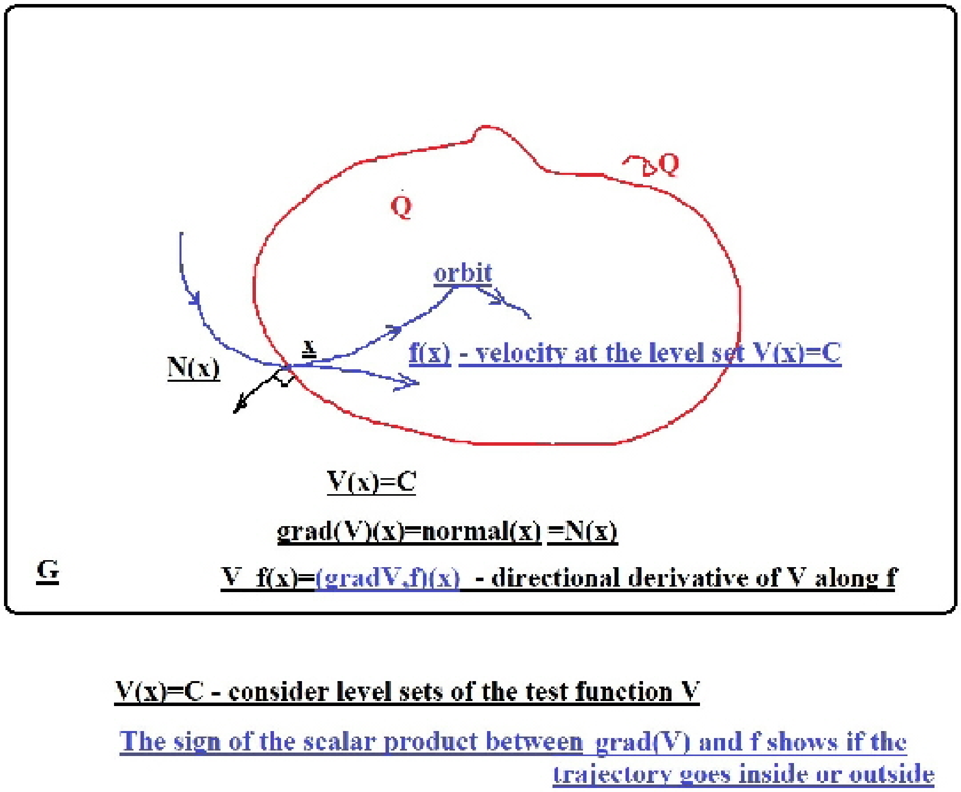
We find a test function $V(x)$ that has some simple level sets $\partial Q = \{x : V(x) = C\}$ that are closed curves (or surfaces in higher dimensions) enclosing a bounded domain Q in G.

Typical examples are $V(x, y) = x^2 + y^2 {=} R^2$ - circle or radius R, or $V(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} {=} 1$ - ellipse, et.c.

• Show that a particular level set ∂Q bounds a positively - invariant set Q we check the sign of the directional derivative of V along the velocity in the equation: $V_f(x) = (\nabla V \cdot f)(x)$ for all points on the level set $\{V(x) = C\}$ for a particular constant C.

• $\nabla V(x)$ is a normal vector to the level set of V that goes through the point x.Therefore the sign of $V_f(x) = (\nabla V \cdot f)(x)$ shows if trajectories go to the same side of the level set as the gradient ∇V (if $V_f(x) > 0$) or to the opposite side (if $V_f(x) < 0$).

• If all trajectories go inside a bounded set Q, then all trajectories starting inside Q will stay inside Q forever.



Example.

Consider the following system of ODEs: $\begin{cases} x' = & 2y \\ y' = & -x-(1-x^2)y \end{cases}$.

Find a compact around the origin that no trajectories escape.

Solution.

We try the test function $V(x,y) = x^2 + 2y^2$ that leads to cancellation of indefinite terms in the directional derivative along trajectories:

$$\begin{aligned} V_f(x,y) &= \frac{d}{dt}V(x(t),y(t)) = \nabla V(x,y)\cdot\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \\ &= \overrightarrow{N}\cdot\overrightarrow{f} = \\ &= \begin{bmatrix} 2x \\ 4y \end{bmatrix}\begin{bmatrix} 2y \\ -x-(1-x^2)y \end{bmatrix} \\ &= -4y^2(1-x^2) \le 0 \end{aligned}$$

$\nabla V(x, y)$ is a normal vector to level sets of the form:

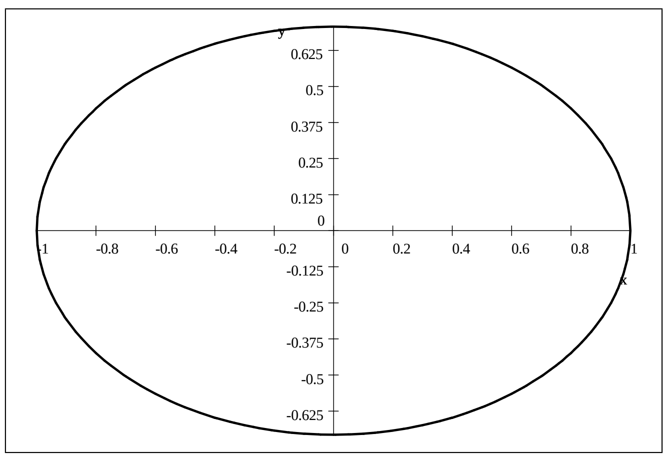
$$x^2 + 2y^2 = C$$

$V_f(x, y) = 4xy - 4xy - 4y^2(1 - x^2) = -4y^2(1 - x^2) \le 0$ that is not positive for $|x| \le 1$.

We see that trajectories of the system will enter the level set of the function $V(x)$ if $|x| \le 1$, namely for points inside the stripe $|x| \le 1$ in the plane. Level sets of $V(x) = C$ are ellipses oriented along coorinate axes. The largest one inside the stripe $|x| \le 1$ must go through the point $(1, 0)$. We choose corresponding value of the constant C in the equation for this leel set.

We put $y = 0$, $x = 1$, into te equation $x^2 + 2y^2 = C$ and conclude that $C = 1$. The desired level set is $x^2 + 2y^2 = 1$.

Trajectories starting inside the compact bounded by this ellips stay inside it forever.



■

$$x'(t) = f(t, x), \quad f : J \times G \to \mathbb{R}^n; \quad x(\tau) = \xi \tag{4}$$

with $J \subset \mathbb{R}$ - an interval, $G \subset \mathbb{R}^n$, open, $\tau \in J$, $\xi \in G$, f- continuous in $J \times G$.The set $J \times G$ is a "cylinder" in $\mathbb{R}^{n+1}$ with the bottom G.

The following Theorem describes the situation in a sense opposite to the previous Corollary 4.10. It describes the behaviour of maximal solutions to the I.V.P. above having bounded maximal interval $I_{\max}$ (if J is $\mathbb{R}$), or in the case when the interval J has bounded endpoints itself, describes maximal solutions with maximal interval that is "bounded with respect to J", meaning that $\sup I_{\max} < \sup J$ or $\inf J < \inf I_{\max}$.

Theorem 4.11, p.112. "Short living" maximal solutions escape any compact.

Let $x : I \to G$ be a maximal solution to (1) with maximal interval of existence $I \subset J$ and assume that I is not the whole J: $I \neq J$. Denote endpoints of I as $\alpha = \inf(I)$, $\omega = \sup(I)$.Then one of endpoints does not belong to I:

1) $\omega \in J \setminus I$

or

2) $\alpha \in J \setminus I$.

Statement of the Theorem:

1) In the first case $\omega \in J \setminus I$ for each compact $C \subset G$, there is an "escaping time moment" $\sigma \in I$, $\sigma < \omega$, such that $x(t)$ "escapes" C at time σ and never comes back: $x(t) \notin C$ for all $t \in (\sigma, \omega)$.

This property can be further geometrically specified. If $G \neq \mathbb{R}^n$ the trajectory $x(t)$ tends to the boundary ∂G of G with $t \to \omega$ (if G is bounded). It can also tend to infinity if G has "branches" going to infinity in $\mathbb{R}^n$. If $G = \mathbb{R}^n$, then $\|x(t)\| \to \infty, \quad as\ t \to \omega$. This statement is formulated formally as:

$$\begin{aligned} \lim_{t \to \omega} \min \{\operatorname{dist}(x(t), \partial G), 1/\|x(t)\|\} &= 0, \quad for\ G \neq \mathbb{R}^n \\ \|x(t)\| &\to \infty, \quad as\ t \to \omega, \quad for\ G = \mathbb{R}^n \end{aligned} \tag{5}$$

2) Similar statements are valid for limits of $x(t)$ as $t \to \alpha$ for the maximal solution having maximal interval with the left end point α "in the past" belonging to J.

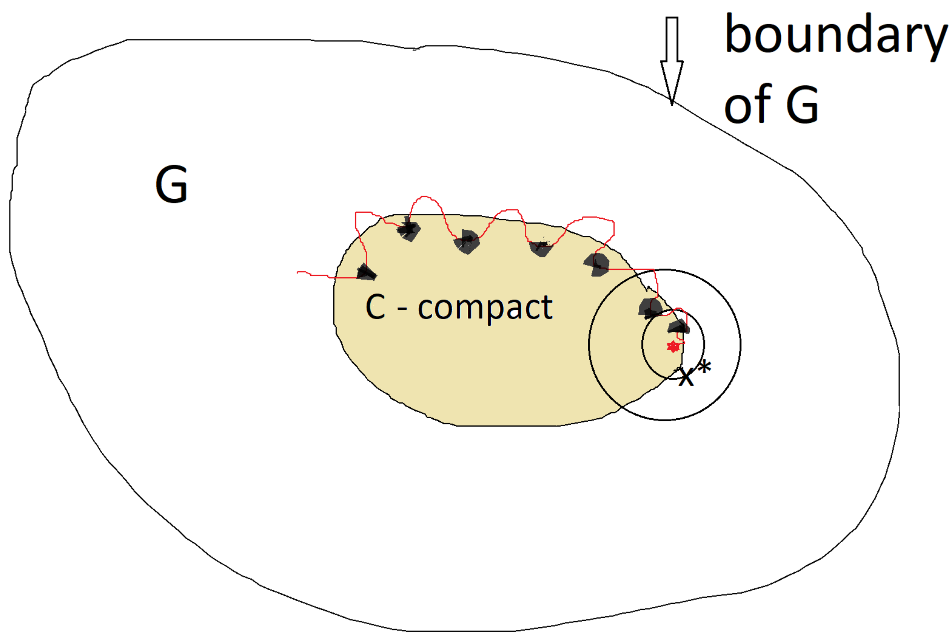
Proof. (the proof in the course book is incomplete)

We consider the case 1). The fact that the maximal solution must at some time leave any compact C follows from the previous Corollary 4.10 by contradiction, because a solution that stays in a compact must have a maximal interval infinite to the right or $[\tau, \infty) \cap I = [\tau, \infty) \cap J$. It contradicts to the condition that $\omega \in J \setminus I$ that means that the given maximal $x(t)$ solution does not reach the maximal possible time in J.

A more sophisticated argument shows that in our situation the solution $x(t)$ must at some time σ leave any compact C "forever". There is a "last visit" time $\sigma < \omega$, such that $x(t)$ never enters C again after this time.

Suppose the opposite, namely that there is a monotone sequence of times $\{t_m\}_{m=1}^{\infty}$ such that $t_m \nearrow \omega$ with $m \to \infty$ such that $x(t_m) \in C$.

C is a compact, therefore there must exist a subsequence (for which we will **keep the same notation** $\{t_m\}_{m=1}^{\infty}$), such that with $m \to \infty$ $t_m \nearrow \omega$ and $x(t_m) \to x_* \in C$.



Choose an r so small that the ball $B((\omega, x_*), r)$ with the center (ω, x_*) would belong to the domain of the equation: $B((\omega, x_*), r) \subset J \times G$. Choose a smaller ball $B \equiv B((\omega, x_*), 2\varepsilon)$ with $\varepsilon = r/3$. Then the closure $\overline{B}$ of B also belongs to the domain of the equation: $\overline{B} \subset J \times G$.

Denote by $M = \sup \left\{ \|f(t,x)\| : (t,x) \in \overline{B} \right\}$ the supremum of the continuous function $\|f(t,x)\|$ on the compact $\overline{B}$.

Using that $t_m \nearrow \omega$, and the boundedness of $\|f(t,x)\| < M$ on $\overline{B}$, we will observe that the index m can be chosen so large that the trajectory $\{(t, x(t) : t \in [t_m, \omega)\}$ of the solution $x(t)$ for $t \in [t_m, \omega)$, on the short time interval $[t_m, \omega,)$ belongs to $\overline{B}$.

It can be observed by considering the integral form of the differential equation and using the estimate $M = \sup \left\{\|f(t,x)\| : (t,x) \in \overline{B}\right\}$ for f on $\overline{B}$:

$$
\begin{aligned}
x(t) &= x(t_m) + \int_{t_m}^{t} f(s, x(s))ds \\
x(t) - x_* &= x(t_m) - x_* + \int_{t_m}^{t} f(s, x(s))ds \\
\|x(t) - x_*\| &\leq \|x(t_m) - x_*\| + |t - t_m| \, M \\
&\leq \varepsilon, \quad m > m_*
\end{aligned}
$$

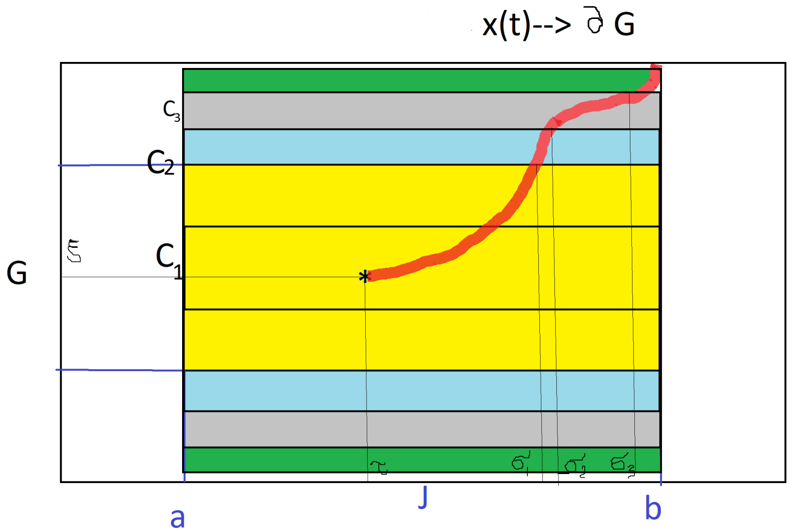
where $\|x(t_m) - x_*\| \to 0$ with $m \to \infty$, and for $t \in [t_m, \omega)$ we have $|t - t_m| \, M \leq |\omega - t_m| \, M \to 0$ with $m \to \infty$. We can choose $m > m_*$ so large that the right hand side in the inequality will be smaller than ε.

Therefore $\|x(t) - x_*\| \leq \varepsilon$, $|t - t_m| \leq \varepsilon$ and the trajectory $\{(t, x(t) : t \in [t_m, \omega)\}$ belongs to $B \equiv B((\omega, x_*), 2\varepsilon)$ and is bounded.

Therefore the closure of the orbit $\{x(t) : t \in [t_m, \omega)\}$ is compact and belongs to G.

Therefore according to the Lemma 4.9 the solution $x(t)$ can be extended up to the time ω and also beyond it, to an even larger time interval $[t_m, \omega + \delta)$. This fact contradicts the given condition that $x(t)$ is the maximal solution with the maximal interval $I_{\max}$ having $\sup I_{\max} = \omega$.

The property that $x(t)$ tends to the boundary of G can be shown in the following way.



If G is bounded, one can construct a rising sequence of compact sets $\{C_n\}_{n=1}^{\infty}$, $C_n \subset C_{n+1} \subset G$ like "blowing up ballons" tending to the boundary ∂G of G so that $\operatorname{dist}(C_n, \partial G) \to 0$ as $n \to \infty$. For each of these sets there is a time σ_n such that $x(t)$ leaves C_n and therefore has $\operatorname{dist}(x(t), \partial G) < \operatorname{dist}(C_n, \partial G)$ for $t > \sigma_n$. This construction proves the fact that $\operatorname{dist}(x(t), \partial G) \to 0$ as $t \to \omega$.

In the case of $G = \mathbb{R}^n$ one can choose a sequence of test compact sets $\{C_n\}_{n=1}^{\infty}$ as balls with centers in the origin and radii r_n tending to infinity with $n \to \infty$ leading together with the "escaping property" to conclusion that $\|x(t)\| \to \infty, \quad as\ t \to \omega$.

The third case with unbounded G with non-empty boundary ∂G can be proven by a combination of the above arguments.■

Proposition 4.12, p. 114 on "eternal" solutions for equations with linear bound for the right hand side. (proof required at exam)

Consider the initial value problem

$$x'(t) = f(t, x(t)), \quad x(\tau) = \xi$$

where $f : J \times \mathbb{R}^N \to \mathbb{R}^N$, continuous and locally Lipschitz in x.

Assume that for any compact interval $K \subset J$ there is $L > 0$ such that for $t \in K$ the following estimate holds for the right hand side:

$$\|f(t,x)\| \leq L(1+\|x\|). \tag{6}$$

If $x : I \to \mathbb{R}^N$ is a maximal solution to the equation $x'(t) = f(t, x(t))$, then $I = J$. In particular if $J = \mathbb{R}$, the maximal solution is defined for all t.

Proof.

Define $\omega = \sup I$, $\alpha = \inf I$. We use proof by contradiction. Suppose that the statement of the theorem is not true, for example that $\omega \in J$ and $\omega \notin I$ and that $\tau < \omega$.

Let choose the constant L such that the (6) for f is valid for $t \in [\tau, \omega]$.Then, using the integral form of the I.V.P. and the triangle inequality implies the following estimate

$$\begin{aligned} \|x(t)\| &\leq \|x(\tau)\| + \int_\tau^t \|f(s,x(s))\|\,ds \leq \|x(\tau)\| + L\int_\tau^t (1+\|x(s)\|)\,ds \\ &= \|x(\tau)\| + L(t-\tau) + \int_\tau^t L\,\|x(s)\|\,ds \end{aligned}$$

for all $t \in [\tau, \omega)$.

The Grönvall inequality implies that $\|x(t)\|$ bounded by a constant C for t on $[\tau, \omega)$. It makes that the corresponding orbit $\{x(t), t \in [\tau, \omega)\}$ is bounded and therefore has a compact closure in $\mathbb{R}^N$. The Lemma 4.9 implies that the solution can be extended to the closed interval $[\tau, \omega]$ and actually by the existence theorem to an even larger interval beyond ω. It contradicts to the supposition that I is a maximal interval for $x(t)$.■

Proof for the case when $\alpha \in J$ and $\alpha \notin I$, $\alpha < \tau$ is treated similarly.

Example. $f(x) = \sin(x^2+t)\frac{(3x^2+t)}{(1+|x|)}$, $t \in (0, \infty)$ satisfies conditions in the theorem.

0.3 Transition map

Existence theorems by Picard and Lindelöf (Theorems 4.17 and 4.22) imply that for any point $\tau, \xi \in J \times G$ there is a unique maximal solution that is convenient to consider as a function

$$x(t) = \varphi(t, \tau, \xi) : J \times J \times G \to G$$

of three variables equal to the maximal solution x of (1). It is a common situation in applications that one is interested not in properties of one solution, but in a description as a whole of the family of solutions with all possible initial data. This type of problems constitute modern theory of differential equations and dynamical systems and motivates introducing the following notion.

Definition. p. 126. Transition map. The mapping $\varphi(t, \tau, \xi)$ defined above is called **transition map.**

Transition map for autonomous systems. In the case of autonomous systems there is no meaning in considering different initial times τ, because all solutions are functions of the time shift $t - \tau$. In this case we consider transition mappings $\varphi(t, \xi) : J \times G \to G$ with

$$\varphi(t, \xi) = x(t), \quad \varphi(0, \xi) = \xi$$

being the maximal solution of (2) with initial condition $x(0) = \xi$.

Local flow or local dynamical system corresponding to an autonomous system of differential equations.

In the modern theory of ODE and dynamical systems the mapping $\varphi(t, \xi)$ corresponding an autonomous differential equation is often called the **local flow or the local dynamical system** corresponding to the differential equation.

Notation

If the maximal interval I_ξ corresponding to the initial point ξ coinsides with $\mathbb{R}$ we say that the solution $\varphi(t, \xi)$ is global. If $I_\xi = \mathbb{R}$ for all $\xi \in G$ then $\varphi(t, \xi)$ is said to be a flow or a dynamical system on G.

Example 4.33 of a transition map.

$G = \mathbb{R}$; $f : G \to \mathbb{R}$; $f(x) = x^2$; for $\xi = 0$; $x(t) \equiv 0$.

Initial data $x(0) = \xi$

$$\begin{aligned}
\frac{dx}{dt} &= x^2; \quad \int \frac{dx}{x^2} = \int dt; \\
-\frac{1}{x} &= t + C \\
-\frac{1}{x} &= t - \frac{1}{\xi}; \quad -\frac{1}{x} = \frac{t\xi - 1}{\xi} \\
x(t) &= \frac{\xi}{(1 - t\xi)}
\end{aligned}$$

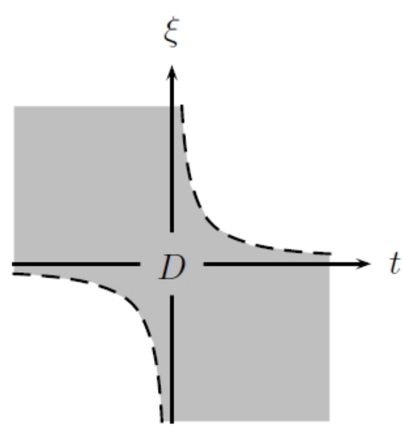
The maximal interval for $\xi = 0;\ \ x(t) \equiv 0.$ is $I_\xi = \mathbb{R}$

The maximal interval for $\xi > 0$, $I_\xi = (-\infty, 1/\xi)$.

The maximal interval for $\xi < 0$, $I_\xi = (1/\xi, \infty)$

$$\varphi(t,\xi) = \frac{\xi}{(1 - t\xi)}; \quad D(\varphi) = \{(t,\xi) \in \mathbb{R} \times \mathbb{R}; \quad t\xi < 1\}$$

The domain D of φ is an open set. The function $\varphi(t,\xi)$ is continuous and even locally Lipschitz.



Proposition. Theorem 4.34, p.139 (consequence of Th. 4.29, p. 129)

The domain $D = \{(t,\xi) \in I_\xi \times G, \quad \xi \in G\}$ of the transition map $\varphi(t,\xi)$ is open and $\varphi(t,\xi)$ is continuous and even locally Lipschitz in D.

Proof of the Lipschitz property with respect to each of the variables t and ξ follows from the integral form of the I.V.P., and for ξ variable - from an application of Grönwall inequality similar to the proof of uniqueness of solutions to I.V.P.

It is a **good exercise** to carry out a proof of this property with respect to one of the variables.

Lecture 16
Summary of Lecture 15

1. Method of test functions for finding a positiely invariant compact set enclosing all trajectories that start inside it .

2. Escaping of compact property for "short living" solutions.

3. "Eternal" solutions are guaranteed for equations with linear bound on the right hand side.

4. Transition mapping $\varphi(t, \xi)$ and it's properties: openess of the domain and local Lipschitz porperty with respect to both variables. Example.

Point out that in the later part of the course we always suppose that conditions of the Picard-Lindelöf theorem are satisfied: the right hand side $f(t, x)$ of the ODE is locally Lipschitz with respect to x.

Proposition. Translation invariance of the transition mapping for autonomous systems

(a non-linear version of the Chapman-Kolmogorov relation) Theorem 4.35, p. 140 -141.

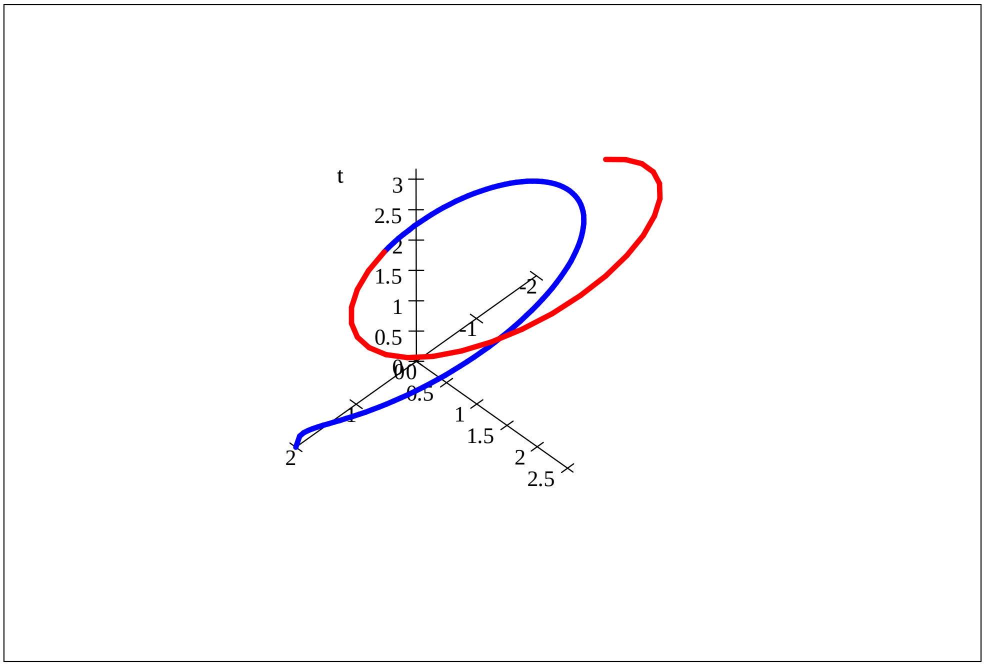
The transition mapping $\varphi(t, \xi)$ for an autonomous ODE has the following properties

(1) $\varphi(0, \xi) = \xi$ for all $\xi \in G$

(2) if $\xi \in G$ and $\tau \in I_\xi = I_{\max}(\xi)$ - maximal interval for ξ, then

$$\begin{aligned} I_{\varphi(\tau,\xi)} &= (I_\xi) - \tau \\ \varphi(t + \tau, \xi) &= \varphi(t, \varphi(\tau, \xi)), \quad \forall t \in I_\xi - \tau \end{aligned}$$

Proof of this statement is similar to the proof of the Chapman Kolmogorov relations for linear systems.



We consider first a trajectory $\varphi(...,\xi)$ starting at the point $\xi \in G$ at time $t = 0$ and finishing at time τ at the point $\varphi(\tau, \xi)$ (blue curve on the picture). Then we continue this movement from the last point $\varphi(\tau, \xi)$ during time t (red curve) coming finally to the point $\varphi(t, \varphi(\tau, \xi))$ in the right hand side of the equation in the conclusion of the theorem.

The fact that solutions are unique (meaning that trajectories have no branches) and the equation is autonomous (velocity field f is independent of time) makes that this movement is equivalent to just moving with the flow starting from the point ξ during the total time $t + \tau$, that is the left hand side in the equation. The illustration here is borrowed from the proof for the linear systems but is otherwise completely abstract. The only difference here is that we have a superposition $\varphi(t, \varphi(\tau, \xi))$ of transfer mappings in the non-linear case instead of the product of transfer matrices in the linear case (which corresponds to a superposition for linear mappings).