FOURIER ANALYSIS & METHODS 2020.03.03

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Solving PDEs with the help of SLPs

We have seen how the process of solving PDEs like the heat and wave equation often leads to a set of functions which comprise an orthogonal basis for $\mathcal{L}^2$ or a weighted $\mathcal{L}^2$ space. These basis functions generally come from separation of variables. When we solve the "space" part of the PDE, we very often end up solving a type of SLP. The easiest examples are:

$$f'' = \lambda f, \quad f(a) = 0 = f(b), \text{ for f defined on the interval, } [a, b]$$
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A more challenging example comes from solving the heat and wave equations on a circular sector. There, when we did separation of variables, we got the nice type of SLP above for the angular variable (θ), and we got a more complicated SLP for the radial variable. This turned into a Bessel equation. We used the initial data to determine the coefficients in our series expansion, by writing the initial data as a Fourier-Bessel type series.

2. The French polynomials

In *other* geometric settings, this same process will lead to *other* special functions. In the last part of this course, based on chapter 6 in Folland, we will look at *the French polynomials,*

(1) Legendre polynomials
(2) Hermite polynomials
(3) Laguerre polynomials

We can imagine that now we may be solving PDEs in more exotic geometric settings, like French Polynesia. Hence, more exotic functions will play the role of the SLP part of the problem. Three such types of functions are the aforementioned French polynomials.

2.1. Legendre polynomials.

These French polynomials arise from using spherical coordinates to solve the wave and heat equations on a three-dimensional sphere.

2.2. **Hermite polynomials.** These French polynomials arise from using parabolic coordinates to solve the wave and heat equations in a parabolic shaped region.

2.3. **Laguerre polynomials.** These French polynomials arise from the quantum mechanics of the hydrogem atom.

2.4. **Orthogonal polynomials general theory.** For the purpose of this course, it is most important that you learn how to *use* the French polynomials. Depending on how much time we have, we may go into the details of the origins of the French polynomials, but these details are rather complicated and will not be examined. So, we prioritize that which shall be examined. The following proposition is therefore useful.

Proposition 1. *Assume that $\{p_n\}_{n\in\mathbb{N}}$ is a sequence of polynomials such that p_n is of degree n for each n. Assume that $p_0 \neq 0$. Then for each $k \in \mathbb{N}$, any polynomial of degree k is a linear combination of $\{p_j\}_{j=0}^k$.*

Proof: The proof is by induction. If q_0 is a polynomial of degree 0, then we may simply write

$$q_0 = \frac{q_0}{p_0} p_0.$$

This is okay because p_0 is degree zero, so it is a constant, and $p_0 \neq 0$, so the coefficient q_0/p_0 is also a constant. Assume that we have verified the proposition for all $0, 1, \ldots k$. We wish to show that it holds for $k+1$. So, let q be a polynomial of degree $k+1$. This means that

$$q(x) = ax^{k+1} + l.o.t. \quad l.o.t. \text{ means lower order terms}$$

has

$$a \neq 0.$$

Moreover, since p_{k+1} is of degree $k+1$ (not of a lower degree), it is of the form

$$p_{k+1} = bx^{k+1} + l.o.t., \quad b \neq 0.$$

So, let us consider

$$q(x) - \frac{a}{b} p_{k+1}(x) = p(x) \text{ which is degree } k.$$

By induction, p is a linear combination of $p_0, \ldots, p_k$. Therefore

$$q(x) = \frac{a}{b} p_{k+1} + \sum_{j=0}^{k} c_j p_j,$$

for some constants $\{c_j\}_{j=0}^k$.

Proposition 2. *Let $\{p_k\}_{k=0}^\infty$ be a set of polynomials such that each p_k is of degree k, and $p_0 \neq 0$. Moreover, assume that they are $\mathcal{L}^2$ orthogonal on a finite bounded interval $[a,b]$. Then these polynomials comprise an orthogonal basis of $\mathcal{L}^2$ on the interval $[a,b]$.*

Proof: Assume that some $f \in \mathcal{L}^2$ on the interval is orthogonal to all of these polynomials. Therefore by the preceding proposition, f is orthogonal to *all* polynomials. To see this, note that if p is a polynomial of degree n, then there exist numbers $c_0, \ldots, c_n$ such that

$$p = \sum_{j=0}^{n} c_j p_j \implies \langle f, p \rangle = \sum_{j=0}^{n} \overline{c_j} \langle f, p_j \rangle = 0.$$

We shall use the fact that continuous functions are dense in $\mathcal{L}^2$. Therefore given $\varepsilon > 0$, there exists a continuous function, g, such that

$$||f - g|| < \frac{\varepsilon}{2(||f|| + 1)}.$$

Next, we use the Stone-Weierstrass Theorem which says that all continuous functions on bounded intervals can be approximated by polynomials. Therefore, there exists a polynomial p such that

$$||g - p|| < \frac{\varepsilon}{2(||f|| + 1)}.$$

Finally, we compute

$$||f||^2 = \langle f, f \rangle = \langle f - g + g - p + p, f \rangle = \langle f - g, f \rangle + \langle g - p, f \rangle + \langle p, f \rangle$$
$$= \langle f - g, f \rangle + \langle g - p, f \rangle.$$

By the Cauchy-Schwarz inequality,

$$||f||^2 \le ||f - g||||f|| + ||g - p||||f|| < \frac{||f||\varepsilon}{2(||f|| + 1)} + \frac{||f||\varepsilon}{2(||f|| + 1)} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows that $||f|| = 0$. Hence by the three equivalent conditions to be an orthogonal basis, we have that the polynomials are an orthogonal basis of $\mathcal{L}^2$ on the interval.

2.5. **Best approximations.** We recall a slight variation of the best approximation theorem:

Theorem 3. *Let $\{\phi_n\}_{n \in \mathbb{N}}$ be an orthonormal set set in a Hilbert space, H. If $f \in H$,*

$$||f - \sum_{n \in \mathbb{N}} \langle f, \phi_n \rangle \phi_n|| \le ||f - \sum_{n \in \mathbb{N}} c_n \phi_n||, \quad \forall \{c_n\}_{n \in \mathbb{N}} \in \ell^2,$$

and $=$ holds $\iff$ $c_n = \langle f, \phi_n \rangle$ holds $\forall n \in \mathbb{N}$. More generally, let $\{\phi_n\}_{n=0}^N$ be an orthogonal, non-zero set in a Hilbert space H. Then,

$$||f - \sum_{n=0}^{N} \frac{\langle f, \phi_n \rangle}{||\phi_n||^2} \phi_n|| \le ||f - \sum_{n=0}^{N} c_n \phi_n||, \quad \forall \{c_n\}_{n=0}^N \in \mathbb{C}^{N+1}.$$

Equality holds if and only if

$$c_n = \frac{\langle f, \phi_n \rangle}{||\phi_n||^2}, \quad n = 0, \ldots, N.$$

How to prove it? The only difference is the last part, but we can use the proof of the first part. Define $\psi_n = 0$ for $n > N$. Next define

$$\psi_n = \frac{\phi_n}{||\phi_n||}, \quad n = 0, \ldots, N.$$

Repeat the argument in the proof of the best approximation theorem using $\{\psi_n\}_{n\in\mathbb{N}}$ instead of ϕ_n.

$$||f - \sum_{n\in\mathbb{N}} c_n\psi_n||^2 = ||f - \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n + \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n - \sum_{n\in\mathbb{N}} c_n\psi_n||^2$$

$$= ||f - \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n||^2 + || \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n - \sum_{n\in\mathbb{N}} c_n\psi_n||^2 + 2\Re\langle f - \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n, \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n - \sum_{n\in\mathbb{N}} c_n\psi_n\rangle.$$

The scalar product

$$\langle f - \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n, \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n - \sum_{n\in\mathbb{N}} c_n\psi_n\rangle = \langle f, \sum_{n\in\mathbb{N}} (\hat{f}_n - c_n)\Psi_n\rangle - \sum_{n\in\mathbb{N}} \hat{f}_n\langle\psi_n, \sum_{m\in\mathbb{N}} (\hat{f}_m - c_m)\Psi_n\rangle.$$

By the orthogonality and definition of Ψ_n, and the definition of $\hat{f}_n$,

$$= \sum_{n\in\mathbb{N}} \hat{f}_n\overline{(\hat{f}_n - c_n)} - \sum_{n\in\mathbb{N}} \hat{f}_n \sum_{m\in\mathbb{N}} \overline{(\hat{f}_m - c_m)}\langle\psi_n, \psi_m\rangle$$

$$= \sum_{n\in\mathbb{N}} \hat{f}_n\overline{(\hat{f}_n - c_n)} - \sum_{n\in\mathbb{N}} \hat{f}_n\overline{(\hat{f}_n - c_n)} = 0.$$

Therefore

$$||f - \sum_{n\in\mathbb{N}} c_n\psi_n||^2 = ||f - \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n||^2 + || \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n - \sum_{n\in\mathbb{N}} c_n\psi_n||^2$$

$$= ||f - \sum_{n=0}^{N} \hat{f}_n\psi_n||^2 + \sum_{n=0}^{N} |\hat{f}_n - c_n|^2 \le ||f - \sum_{n=0}^{N} \hat{f}_n\psi_n||^2,$$

with equality if and only if $c_n = \hat{f}_n$ for all n. Since

$$\sum_{n=0}^{N} \hat{f}_n\psi_n = \sum_{n=0}^{N} \frac{\langle f, \phi_n\rangle}{||\phi_n||^2}\phi_n,$$

this completes the proof.

2.5.1. *Applications: best approximation problems.* This shows us that if we have a finite orthogonal set of non-zero vectors in a Hilbert space, then for any element of that Hilbert space, the best approximation of f in terms of those vectors is given by

$$\sum_{n=0}^{N} \frac{\langle f, \phi_n\rangle}{||\phi_n||^2}\phi_n.$$

Here is the setup of questions which can be solved using this theory. Either:

(1) You are given functions defined on an interval which are $\mathcal{L}^2$ orthogonal on that interval (possibly with respect to a weight function which is also specified). Either you recognize that they orthogonal because you've seen them before (like sines, cosines, from problems you have solved previously) *or* you *compute* that they are $\mathcal{L}^2$ orthogonal on the interval. Then, you are asked to find the numbers $c_0, c_1, \ldots c_N$ so that the $\mathcal{L}^2$ norm, or the weighted $\mathcal{L}^2$ norm of $f - \sum_{k=0}^N c_k \phi_k$ is minimized, where the function f is also specified.
(2) You are asked to find the polyonomial of at most degree N such that the $\mathcal{L}^2$ norm (or weighted $\mathcal{L}^2$ norm) of $f - p$ where p is a polynomial is minimized.

In the first case, you compute

$$c_k = \frac{\langle f, \phi_k \rangle}{||\phi_k||^2}.$$

In the second case you need to *build up a set of orthogonal or orthonormal polynomials.* Then, you let ϕ_k be defined to be the polynomial of degree k you have built. Proceed the same as in the first case, and your answer shall be

$$\sum_{k=0}^N c_k \phi_k.$$

If you don't like the thought of building up a set of orthogonal polynomials, if you are lucky, then it may be possible to suitably modify some of the French polynomials to be orthogonal on the interval under investigation, with respect to the (possibly weighted) $\mathcal{L}^2$ norm. So, we shall proceed to study the French polynomials. Depending on how much time we have, we may also be able to get into their "origin stories."

2.6. **The Legendre polynomials.** The *Legendre polynomials,* are defined to be

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left((x^2 - 1)^n \right).$$

OMG like why on earth are they defined in such a bizarre way, right? What did you expect, they are French polynomials! Of course they are not defined in some simple way, mais non, they must be all fancy and shrouded in mystery and intrigue. Actually though, the reason comes from the PDE in which they arise as solving one part of the separation of variables for the heat and wave equations in three dimensions using spherical coordinates. First, let us do a small calculation involving these polynomials:

$$(x^2 - 1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (x^2)^k = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} x^{2k}.$$

Therefore, if we differentiate n times, only the terms with $k \geq n/2$ survive. Differentiating a term x^{2k} once we get $2kx^{2k-1}$. Differentiating n times gives

$$\frac{d^n}{dx^n}(x^{2k}) = x^{2k-n} \prod_{j=0}^{n-1} (2k - j).$$

If we want to be really persnickety, we prove this by induction. For $n = 1$, we get that

$$(x^{2k})' = 2kx^{2k-1}.$$

Which is correct. If we assume the formula is true for n, then differentiating $n+1$ times using the formula for n we get

$$(2k-n)x^{2k-(n+1)}\prod_{j=0}^{n-1}(2k-j) = x^{2k-(n+1)}\prod_{j=0}^{n}(2k-j).$$

See, it is correct. As a result,

$$P_n(x) = \frac{1}{2^n n!}\sum_{k\geq n/2}^{n}(-1)^{n-k}\binom{n}{k}x^{2k-n}\prod_{j=0}^{n-1}(2k-j).$$

So, we see that this is indeed a polynomial of degree n.

Next time we will prove the following theorem about the Legendre polynomials.

Theorem 4. *The Legendre polynomials are orthogonal in $\mathcal{L}^2(-1,1)$ and*

$$||P_n||^2 = \frac{2}{2n+1}.$$

Here, we shall simply use this theorem to do an example.

Exercise 1. *Find the polynomial $p(x)$ of at most degree four which minimizes the following integral*

$$\int_{-1}^{1}|p(x)-e^x|^2 dx.$$

Based on our theoretical knowledge, the 'best approximation' can be created using the Legendre polynomials. Let

$$f(x) := e^x.$$

Then, the 'best approximation' in terms of the Legendre polynomials is

$$p(x) = \sum_{n=0}^{4} c_n P_n(x),$$

where $P_n(x)$ is the Legendre polynomial of degree n, and

$$c_n := \frac{\langle f, P_n\rangle}{||P_n||^2} = \frac{\int_{-1}^{1} e^x \overline{P_n(x)}dx}{\frac{2}{2n+1}}.$$

The beautiful fact is that we do not need to compute these integrals.

2.7. **Hints for the exercises to be done oneself.**

(1) (5.5.1) A cylinder of radius b is initially at the constant temperature A. Find the temperatures in it at subsequent times if its ends are insulated and its circular surface obeys Newton's law of cooling, $u_r + cu = 0$, $(c > 0)$. Hint: Since the ends are insulated the problem is reduced to a disk. Moreover, since the initial condition is radially symmetric, the solution will also continue to be radially symmetric for all later times. Thus, you just need $u(r,t)$ a function depending on the radius and the time. Write $u(r,t) = R(r)T(t)$ and put into the heat equation remembering to use polar coordinates for the PDE. Solve for R first. Use the boundary condition. There will be J_0s and the λ_ks will come from an equation that you need $J_0(\lambda_k r)$ to satisfy (BC!). Then solve for the time part, and finally get the coefficients using the initial condition.

(2) (5.5.5) Solve the problem

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = 0 \text{ in } D = \{(r,\theta,z) : 0 \le r \le b, 0 \le z \le l\}$$

$$u(r,\theta,0) = 0, \quad u(r,\theta,l) = g(r,\theta), \quad u(b,\theta,z) = 0.$$

Hint: There is only one inhomogeneous part of the equation, and that is the boundary condition when $z = l$. Otherwise, observe that since we are in a cylinder, the function must be 2π periodic in the theta variable. So, let us separate variables writing $u = R(r)\Theta(\theta)Z(z)$. Put this into the PDE. First solve for the theta dependent function. I am guessing you will get either $e^{in\theta}$ for $n \in \mathbb{Z}$ or $\sin(n\theta)$ and $\cos(n\theta)$, and these are equivalent to each other... Next, I would solve for the R function. This has the zero boundary condition: $R(b) = 0$. So, I am guessing you will get $J_n(z_{n,k}r/b)$ where $z_{n,k}$ is the k^{th} positive zero of the Bessel function J_n for $n \in \mathbb{N}$. Last but not least, use these to solve for your Z function. Since the PDE is homogeneous, smash them all together into a super-solution using superposition. Use the condition $u(r,\theta,l) = g(r,\theta)$ to specify what the constants in your solution need to be.

(3) (5.2.4) Demonstrate the identity:

$$\int_0^x sJ_0(s)ds = xJ_1(x), \quad \int_0^x J_1(s)ds = 1 - J_0(x).$$

Hint: Use the recurrence formulas. Integrating by parts is a reasonable idea as well.

(4) (5.5.6) Find the steady-state temperature in the cylinder $0 \le r \le 1$, $0 \le z \le 1$ when the circular surface is insulated, the bottom is kept at temperature 0, and the top is kept at temperature $f(r)$. Hint: This is a radially symmetric problem, so you'll have the variables r, z. No thetas. No t because you're asked to find the 'steady-state temperature' so, this is the temperature that is independent of time. Use separation of variables, writing $u(r,z) = R(r)Z(z)$. The boundary condition for R will be that $R'(1) = 0$, because no heat is lost outside the circular surface. The boundary condition for Z is weird. So, solve for R first. The operator $\partial_{xx} + \partial_{yy} + \partial_{zz}$ in these coordinates is

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}.$$

Since it is steady state, you're solving $\Delta RZ = 0$. Solve for R first. Then use it to solve for Z. This will involve expanding $f(r)$ in a series...

(5) Eö 29 Hint: Oh geez. Look at that boundary condition. It depends on time. Well, let's not panic. This is a new trick. Look at the function $(t+1)$. You want that sitting at $x = 0$, but you want to kill it at $x = 1$. How to achieve this using t and x?

$$(t+1)(1-x).$$

This takes care of the boundary condition at $x = 0$, the boundary condition at $x = 1$, and the initial condition at $t = 0$. Does it screw up the PDE? Well,

$$(\partial_t - 2\partial_{xx})(t+1)(1-x) = 1 - x.$$

So now you've got an inhomogeneous PDE. Use the series technique. First, find the basis

$$X_n \text{ with } X_n(0) = X_n(1) = 0, \quad X_n'' = \lambda_n X_n.$$

Find the lambdas. Next write

$$v(x,t) = \sum_{n\geq 1} T_n(t) X_n(x).$$

Expand $-(1-x)$ in an X_n Fourier series, like

$$-(1-x) = \sum_{n\geq 1} b_n X_n(x).$$

Stick v into the PDE. Set it equal to the series for $-(1-x)$. Use the differential equation satisfied by X_n. Equate the coefficients of X_n on the left and right. This will give an ODE for T_n. Use as initial condition $T_n(0) = 0$. Your full solution will be

$$(t+1)(1-x) + v(x,t).$$

Check that it satisfies everything required. If you're stuck, go back to the first exercise demonstrated on Monday's big group session for inspiration! Also, it might make you feel better to know that I first tried doing some Laplace transform business with this, and it became horrible. Realized that it was so complicated, there must be a better way. Indeed there is.

(6) Eö 35 (sorry forgot this one before) Hint: Since you're in a cylinder, use polar coordinates for x and y, but keep z just as it is. The PDE is therefore

$$(\partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz})u = 0.$$

The function should vanish at $z = 0$ and $z = L$. It's got a strange boundary condition at $r = R$. It might be good to change this R into a ρ in case you'd like to use separation of variables. Try to solve the problem using separation of variables. Solve for Z first. Since the boundary data doesn't depend on θ but only depends on z, the solution is independent of θ. So you're just going to have Z and R. You'll get the coefficients from the boundary data, which might look weird, but should read

$$u = \sin\left(\frac{\pi z}{L}\right) - \sin\left(\frac{\pi z}{L}\right)\cos\left(\frac{\pi z}{L}\right).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).