Biomathematics: A course on some applications of dynamical systems

Equilibrium solutions to reaction-diffusion equations

Anders Källén
MatematikCentrum
LTH
anderskallen@gmail.com
**Introduction**

Consider a scalar equation of the form $u' = f(u)$, where $f$ describes some kind of reaction equation. We know that $u(t) \to u^*$ where $u^*$ is either $\pm \infty$ or some equilibrium to $f$, $f(u^*) = 0$. Which equilibrium we approach depends on the initial value of $u$. If we assume that this equation occurs at each point along the real axis, with different initial values $u_0(x)$ in different points $x$, then the solution is given by a function $u(x,t)$ such that

$$\partial_t u(x,t) = f(u(x,t)), \quad u(x,0) = u_0(x).$$

When $t \to \infty$, $u(x,t)$ will approach a function whose values are different equilibria to $f$, and which these are depends on $u_0$. It means that even if $u_0$ was a smooth function, $u(x,t)$ will approach a step function.

But this is basically a very unrealistic situation. A substance that is spread over a region will not stay put, but will diffuse around. Without the reaction term we would have that $u$ fulfills the diffusion equation. In our situation both things happen, which means we have a so-called reaction-diffusion equation

$$\partial_t u(x,t) = D \partial^2_{xx} u(x,t) + f(u(x,t)), \quad u(x,0) = u_0(x).$$

Here we can always choose the length scale so that $D = 1$, which we do in what follows.

In this chapter we will discuss around what we can say about equilibrium solutions to such equations. More specifically we will study the problem when the differential equation is valid in the interval $0 < x < L$, but that $u = 0$ outside it. It means the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0.$$

Of special interest is the case when $u$ is a population of some pest in an area, but cannot survive outside this area. We might for example spray the outside with some pesticide. The question then is what happens inside the region, and how that depends on the size of the area.

In order to come to the more interesting cases we need to do some preparatory work. In the coming material we always assume that $f(0) = 0$, so that the function $u(x,t) = 0$ for all $x,t$ is a solution of the differential equation with the boundary conditions.

**About steady state solutions and their stability**

An steady state is now not a point, but a function $v(x)$ which is independent of time but fulfills the differential equation with boundary conditions:

$$v''(x) + f(v(x)) = 0, \quad 0 < x < L, \quad v(0) = v(L) = 0.$$

Since we assume that $f(0) = 0$, this problem always has the solution $v(x) = 0$, which we will refer to as the trivial solution. The question is if there are any non-trivial solutions to the problem.

I we find a non-trivial solution $v(x)$, the follow-up question is if the steady state is stable. This means that if $u_0(x) \approx v(x)$, but not equality in all points $x$, will $u(x,t) \to v(x)$ when
$t \to \infty$? In order to discuss this we write

$$u(x, t) = v(x) + w(x, t)$$

which turns the partial differential equation into

$$\partial_t w(x, t) = v''(x) + \partial_{xx}^2 w(x, t) + f(v(x) + w(x, t)) =$$

$$v''(x) + \partial_{xx}^2 w(x, t) + f(v(x)) + f'(v(x))w(x) + \ldots.$$  

if $w$ is small. Since $v$ is a steady state solution this means that for small $w$ we have that

$$\partial_t w(x, t) = \partial_{xx}^2 w(x, t) + f'(v(x))w(x, t).$$

which is a linear partial differential equation in $w$. If every solution to this with a small $w(x, 0)$ is such that $w(x, t) \to 0$ as $t \to \infty$ we say that the steady state is stable. To determine this is in general not an easy task. An exception is the stability of the trivial solution, which also is of special interest.

**Example 1** In order to investigate the stability of the trivial solution we want to find the behaviour of the solution to

$$\partial_t w(x, t) = \partial_{xx}^2 w(x, t) + kw(x, t), \quad w(0, t) = w(L, t) = 0$$

as $t \to \infty$, when $w(x, 0)$ is small. Here $k = f'(0)$.

We can simplify the problem by observing that the function $z(x, t) = e^{-kt}w(x, t)$ satisfies

$$\partial_t z = -kz + e^{-kt}\partial_t w = -kz + e^{-kt}(\partial_{xx}^2 w + kw) = \partial_{xx}^2 z.$$  

About this equation we know that all solutions of the form $z(x, t) = a(t)b(x)$ are

$$z(x, t) = Ae^{-\mu^2 t} \sin(\mu x), \quad \mu = \frac{n\pi}{L},$$

which means that

$$w(x, t) = e^{kt}a(t)b(x) = Ae^{(k-\mu^2)t} \sin(\mu x), \quad \mu = \frac{n\pi}{L}, \quad n = 1, 2, \ldots.$$  

Here we have that $w(x, t) \to 0$ when $t \to \infty$ if $k < \mu^2$, i.e. when

$$kL^2 < n^2\pi^2.$$  

For the equilibrium to be stable it is required that this is true for all integers $n \geq 1$, which is the case if $k \leq 0$, but also for $k > 0$ if only

$$L < \frac{\pi}{\sqrt{k}}.$$  

It follows that a small population on the interval $[0, L]$ will die out if $L$ is small enough. The exponential growth cannot compensate for the loss of individuals on the boundary.
Steady state solutions for some examples

We will now discuss a graphical method which helps us to decide if the problem

\[ v''(x) + f(v(x)) = 0. \]

has a (non-trivial) solution. First we multiply the equation with \( v' \) in order to get the equivalent equation

\[ \frac{d}{dx}\left(\frac{v'(x)^2}{2} + F(v(x))\right) = 0, \]

where \( F \) is a primitive function to \( f \). From this it follows that there is a constant \( E \) such that

\[ \frac{v'(x)^2}{2} + F(v(x)) = E. \]

Conversely, if \( v \) solves this problem for some constant \( E \), it will also solve the original equation. So (1) is equivalent to

\[ v'(x) = \pm \sqrt{2(E - F(v(x)))}. \]

From this we first note that \( x_0 \) is a stationary point to \( v(x) \) if and only if \( F(v(x_0)) = E \).

In order to extract more information, let us consider some simpler examples.

**Example 2** We start with the classical equation

\[ v'' + v = 0. \]

We know its general solution is \( v(x) = A\sin(x + \phi) \) for constants \( A, \phi \). Ignore this knowledge for the moment and instead note that we have \( f(v) = v \), and therefore \( (e.g.) \ F(v) = v^2/2 \). In the plot below the \( F \)‘s graph is shown, together with the piece of the line \( v = E \) that lies above the curve, i.e. the part where \( E - F(v(x)) \geq 0 \).

Corresponding to \( E \) there is a solution with values in \([-\sqrt{2E}, \sqrt{2E}] \). One way to see that is to note that \( v(x) = A\sin(x + \phi) \) is such that

\[ \frac{v'(x)^2}{2} + \frac{v(x)^2}{2} = \frac{A^2}{2}, \]

which means it solves the equation with \( E = A^2/2 \).
But we can also argue as follows. Assume we start in a point \( x_0 \) where \( v'(x_0) > 0 \) with a value \( v(x) < E \). Then \( v'(x) = \sqrt{2E - v(x)^2} > 0 \) until \( x \) reaches a stationary point \( x_1 \). Since \( v''(x_1) = -v(x_1) \neq 0 \) this is not an inflexion point, which means it must be a local maximum. When we passed \( x_1 \) the derivative will therefore be negative and consequently \( v'(x) = -\sqrt{2E - v(x)^2} \). We next follow \( v \) downwards until we reach a local minimum, after which the function again ascends. We see that the solution corresponding to this line must be a periodic solution oscillating between \( \pm \sqrt{2E} \). That it is a sinus function we cannot deduce, of course!

The last argument above can be generalized. Before we do that, let us take another example.

**Example 3** A stationary solution to Fishers equation should satisfy

\[
v'' + v(1 - v) = 0.
\]

Here \( f(v) = v(1 - v) \), so \( F(v) = \frac{v^2}{2} - \frac{v^3}{3} \), whose graph is shown below.

Again we have plotted a number of horizontal line segments that lie above the graph. We will see that each of them corresponds to a solution to the equation.

Each line segment in the plot extends to \(-\infty\) and the point in which they meet \( F \)'s graph is either a maximum or a minimum. So repeating the argument from the last example we see that a solution \( v(x) \) comes from \(-\infty\), turns around in a global maximum and goes back to \(-\infty\).

We can also draw lines above the graph that are to the right of graph, corresponding to functions that have a global minima when they touch the graph.

There is one further option, not shown, a horizontal line above the graph, which corresponds to a function going monotonously from \(-\infty\) to \(\infty\), or the reverse.

We need one further observation. If \( v(x) \) is a solution of \( v'' + f(v) = 0 \), so are also the
functions $x \to v(x + a)$ and $x \to v(a - x)$, for arbitrary constants $a$. Every function will therefore define what is called an equivalence class of functions. Each of the functions in an equivalence class correspond to the same line segment in the examples above. And every line segment corresponds to an equivalence class. This should be kept in mind in the discussion that follows.

We now want to use this to see when the equation $v'' + f(v) = 0$ has a positive (in $(0, L)$) solution such that $v(0) = v(L) = 0$.

**Example 4** We are looking for a positive function $v(x)$ such that

$$v'' + v = 0, \quad v(0) = v(L) = 0.$$ 

By symmetry such a function must have a global maximum $v_m$ in $x = L/2$. In $(0, L/2)$ we then have that

$$v'(x) = \sqrt{2(E - v(x)^2/2)},$$

which is a separable ODE:

$$\int \frac{dv}{\sqrt{2(E - v^2/2)}} = \int dx.$$ 

To integrate along the $x$-axis from 0 to $L/2$ is the same as integrate along the $v$-axis from 0 to $v_m = v(L/2)$, so we have that

$$\int_0^{v_m} \frac{dv}{\sqrt{2(E - v^2/2)}} = \frac{L}{2}.$$ 

But $E = v_m^2/2$, so we can rewrite this as

$$\frac{L}{2} = \int_0^{v_m} \frac{dv}{\sqrt{v_m^2 - v^2}} = \int_0^1 \frac{dz}{\sqrt{1 - z^2}} = \frac{\pi}{2}.$$ 

Here we have made the change of variable $v = v_m z$. The conclusion is that

$$L > \pi$$

for there to be a positive solution to the problem.

Let us extract the most important parts of this discussion, put in a more general context. We are looking for a positive solution in the interval $(0, L)$ to the boundary value problem

$$v'' + f(v) = 0, \quad v(0) = v(L) = 0.$$ 

For symmetry reasons such a solution must have its maximum $v_m$ in the point $x = L/2$ and then $E = F(v_m)$. In the interval $(0, L/2)$ we then have that

$$v'(x) = \sqrt{2(F(v_m) - F(v(x))},$$
since we can solve this by separating variables and integrate from 0 to\( L/2 \) (and 0 to\( v_m \), respectively). Thus we have the relationship

\[
L = \sqrt{2} \int_0^{v_m} \frac{dv}{\sqrt{F(v_m) - F(v)}}.
\]

The integral on the right hand side usually depends on\( v_m \), so we have a relationship between\( L \) and\( v_m \) which must be fulfilled for there to be a positive solution to the problem.

**Example 5** For Fisher’s equation, with\( F(v) = v^2/2 - v^3/3 \), we get

\[
L = \sqrt{2} \int_0^{v_m} \frac{dv}{\sqrt{v_m^2/2 - v_m^3/3 - (v^2/2 - v^3/3)}} = 2 \int_0^1 \frac{dz}{\sqrt{1 - z^2 - 2v_m^3(1 - z^3)}}.
\]

In the graph below we have computed the integral numerically for different\( v_m \), in order to see how\( L \) depends on\( v_m \).

We see that\( L \) is an increasing function of\( v_m \) and that\( L \to \pi \) as\( v_m \to 0 \). We therefore only have non-trivial solutions when\( L > \pi \). Furthermore,\( v_m \to 1 \) as\( L \to \infty \), which simply means that for an infinitely large area the steady state is one everywhere, as it should be.

The graph above is probably more interesting in its inverse form: it tells us how large the maximal value of\( v \) is for an interval of given width\( L \).

**A general theorem**

With notations from above, let\( S \) be the set of horizontal line segments (of positive length) in the\( yz \)-plane such that

a) the segment is strictly above the graph\( z = F(y) \), except at any endpoints,

b) an endpoint of the segment is on the graph.
The graph below shows some examples of elements in the set $S$.

We now have the following theorem

**Theorem 1**

There is a 1-1-relationship between segments in $S$ and equivalence classes of non-constant solutions to (1). The relation is that the segment corresponds to the image of the solution.

The proof gives more information on what the solutions look like, for which typical representatives are given in the graph above. What is true for these solutions are:

(a) This segment corresponds to a function $v$ which increases from $-\infty$ and asymptotically approaches the value that corresponds to the segments right end.

(b) This segment corresponds to a function $v$ which increases from $-\infty$, has a global maximum in a point $x_0$ and then decreases to $-\infty$. The graph of the function is symmetric around the axis $x = x_0$.

(c) This is a function which instead has a global minimum, and goes to infinity on each side of it. Again the function is symmetric around the value giving the minimal value.

(d) This is a function that asymptotically approach 0 as $|x| \to \infty$ and has a global maximum.

(e) This is a periodic solution.

**Remark** Note the segments in (a) and (d): they have one endpoint in a local extreme value for the function $F$. Such a point is not reached in finite time. Also note that a solution corresponding to the segment in e.g. (a) also can be decreasing. Such
functions are part of the equivalence class.
As further preparation for the proof let us draw the corresponding phase space portrait, i.e. the curves \((v, v')\) when \(v\) solves (2) for different \(E\):

![Graph showing phase space portrait with curves labeled (a), (b), (c), (d), and (e)](image)

In the graph we can note a few things:

a) The two curves labelled (a) correspond to two different functions: the upper to an increasing one starting and returning to \(-\infty\), the lower to a decreasing one starting and returning to \(\infty\). But these are members of the same equivalence class!

b) The three blue points represent constant solutions to the equation, i.e. solutions to \(f(u) = 0\). This means that when a curve approaches such a point, it cannot actually reach it. For example, the upper curve labelled (a) approaches the equilibrium to the right. If we call it \((\alpha, 0)\) this means that the corresponding function \(v(x)\) is such that \(v(x) \to \alpha\) when \(x \to \infty\).

c) Let \(W(x)\) be a function corresponding to curve (d). If we normalize it so that it attains its maximum \(\beta\) in the point \(x = 0\), we have an even function such that

\[
W(0) = \beta, \quad \lim_{|x| \to \infty} W(x) = 0
\]

and \(W\) is monoton on each of the half axis \(x < 0\) and \(x > 0\).

We now prove the theorem.

**Bevis.** Given a segment in \(S\), let \(E\) be its level and \(I\) the interval on the \(y\)-axis below the segment. If we integrate the equation

\[
y' = \sqrt{2(E - F(y))}, \quad y(0) = y_0,
\]

we get a strictly increasing solution. Let \((a, b), a < 0 < b\) be the maximal interval on which \(y\) exists. Then one of two cases must occur on the right endpoint \(b\):

a) \(\lim_{x \to b} y(x) = \infty\),
b) \( \lim_{x \to b} y'(x) = 0 \) och \( \lim_{x \to b} F(y(x)) = E \).

The first occurs precisely when \( F(y) < E \) for all \( y > y_0 \), which in turn means that the corresponding segment goes all the way to infinity. In the second case we have that the range of \( y \) has as its upper limit the first value \( y_m \) for which \( F(y_m) = E \). A corresponding analysis is of course true for the other endpoint \( a \) and we see that the range of \( y \) is precisely \( I \).

In the second case with \( b < \infty \) we can continue the solution \( y \) past the endpoint \( b \) an even function w.r.t. \( b \), i.e. as \( y(x) = y(2b - x) \). Also this extended function is a solution of (1), and \( y \) has a local maximum in the point \( x = b \). In the other end, if \( a > -\infty \) and \( y \) is bounded there, we can similarly continue the function to the other side of \( a \), but now with a local minimum in \( x = a \). We see that in this case we get a periodic solution. In summary, every segment in \( S \) corresponds to a solution of (1) whose range is \( S \).

It remains to prove the converse, that each non-constant solution corresponds to a segment in \( S \). For this, let \( y \) be a non-constant solution to (1) and let \( y_0 \) be in the interior of its range. Corresponding to \( y \) there is a constant \( E \) in (2) and we have that \( F(y) \leq E \) for \( y \in I \), and that \( F(y) = E \) when \( y \) attains a local extreme value, i.e. for \( y \) at each finite endpoint of \( I \). It follows that \( I \) is precisely the interval on the \( y \)-axis that contains \( y_0 \) and which corresponds to segments on level \( E \) with finite endpoints (if there are any) on the curve \( F(y) = E \). It remains to be shown that \( F(y) < E \) except at endpoints.

Assume there is an interior value \( y_1 \) in \( I \) such that \( F(y_1) = E \) and \( F'(y_1) = f(y_1) = 0 \). If we integrate (2) we get

\[
x - a = \pm \int_{y(a)}^{y(x)} \frac{dy}{\sqrt{2(E - F(y)))}}.
\]

Close to \( y = y_1 \) we have that \( F(y) = F(y_1) + F'(y_1)(y - y_1) + F''(y_1)(y - y_1)^2/2 + \ldots = F''(y_1)(y - y_1)^2/2 + \ldots \), which means that

\[
\lim_{y \to y_1} \frac{1/\sqrt{2(E - F(y))}}{1/(y_1 - y)} = \frac{1}{\sqrt{F''(y_1)}}
\]

from which we can deduce that the integral diverges when \( y(x) \to y_1 \). (We assume that \( F''(y_1) \neq 0 \). If this is not the case, the integral becomes even more divergent!) In other words, if \( y(x) \to y_1 \), we must have \( x \to \pm \infty \), so \( y \) cannot attain the value \( y_1 \) in the interior of its range. That completes the proof. \( \square \)