Corona risks

Financial Risk
4-th quarter 2020/21
Lecture2: Peaks
over thresholds

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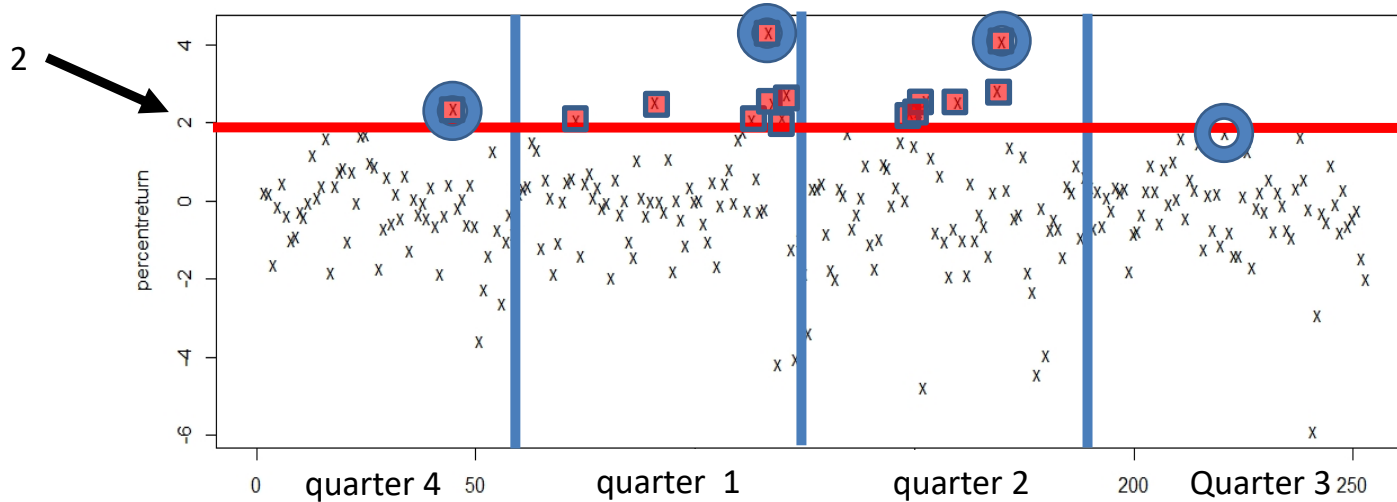www.glasbergen.com

The big recession 2009

"As an alternative to the traditional 30-year mortgage, we also offer an interest-only mortgage, balloon mortgage, reverse mortgage, upside down mortgage, inside out mortgage, loop-de-loop mortgage, and the spinning double axel mortgage with a triple lutz."

Windstorm insurance

Gudrun January 2005
326 MEuro loss
72 % due to forest losses
4 times larger than second largest

1

Apple losses (= - $100 \times \frac{\text{price tomorrow} - \text{price today}}{\text{price today}}$) one year back
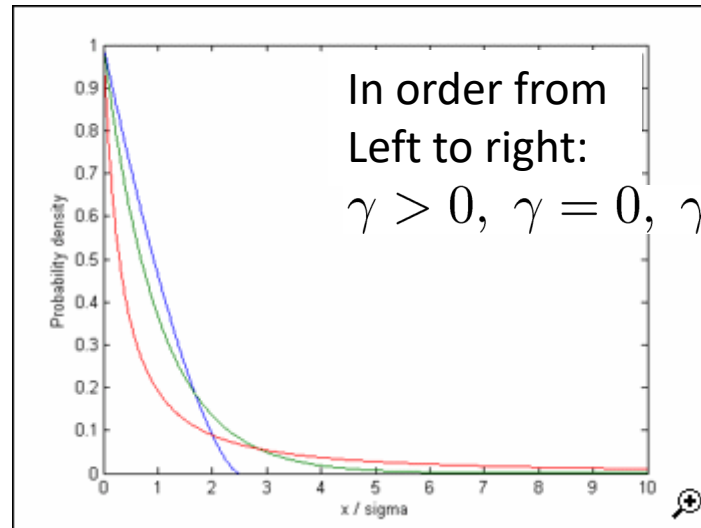

2

percent return

| 0 | quarter 4 | 50 | quarter 1 | quarter 2 | 200 | Quarter 3 | 250 |

⭕ Maximum quarterly loss   🟥 excess of the level $u = 2$

How large is the risk of a big quarterly loss? BM
How large is the risk of a big loss tomorrow? PoT

2

The GP distribution: $H(x) = 1 - \left(1 + \frac{\gamma}{\sigma}x\right)_+^{-1/\gamma}$

In order from Left to right: $\gamma > 0, \ \gamma = 0, \ \gamma < 0,$

density function of Generalized Pareto distribution

$$h(x) = \frac{d}{dx}H(x) = \frac{1}{\sigma}\left(1 + \frac{\gamma}{\sigma}x\right)_+^{-1/\gamma - 1} \quad \left(= \frac{1}{\sigma}e^{-x/\sigma} \ \text{if} \ \gamma = 0\right)$$

$\gamma \geq 0$ the distribution has left endpoint *0* and right endpoint ∞

$\gamma < 0$ the distribution has left endpoint *0* and right endpoint $\sigma/|\gamma|$

the distribution is "heavytailed" for $\gamma > 0$. Then moments of order greater than $1/\gamma$ are infinite/don't exist, exactly as for the Generalized Extreme Value distribution

3

- Peaks over thresholds (PoT) method *(Coles p. 74-91)*

- Choose (high) threshold u and from i.i.d observations $Y_1, \dots, Y_n \sim F$ obtain N threshold excesses $X_1 = Y_{t_1} - u, \dots, X_N = Y_{t_N} - u$, where $t_1, \dots, t_N$ are the times of threshold exceedance

- Assume $X_1, \dots, X_N$ are i.i.d and GP distributed and that $t_1, \dots, t_N$ are the occurrence times of an independent Poisson process, so that N has a Poisson distribution

- Use $X_1, \dots, X_n$ to estimate the GP parameters and N to estimate the mean of the Poisson distribution

- Estimate tail $\bar{F}(x) = 1 - F(x) = \bar{F}(u)\bar{F}_u(x - u)$, where $\bar{F}_u(x - u) =$ the conditional distribution function of threshold excesses, by

$$\hat{\bar{F}}(x) = \frac{N}{n}\ \hat{\bar{F}}_u(x - u)$$

Details:

Assume the random variable X has d.f. F and let u be a (high) level. The distribution of exceedances then is the conditional distribution of $X - u$ given that X is larger than u, *i.e.* it has d.f.

$$F_u(x) = P\big(X - u \leq x \mid X > u\big) = \frac{P\big(X - u \leq x \text{ and } X > u\big)}{P\big(X > u\big)} = \frac{F(x+u) - F(u)}{1 - F(u)}$$

(and hence $\bar{\bar{F}}_u(x) = 1 - F_u(x) = \frac{\bar{F}(x+u)}{\bar{F}(u)}$).

Exercise: Show that if $F(x)$ is a GP distribution, then also $F_u(x)$ is a GP distribution, and express the parameters of $F_u(x)$ in terms of the parameters of $F(x)$ (Treat $\gamma \neq 0$ and $\gamma = 0$ separately.)

More details

N = the (random) number of exceedances of the threshold u by $Y_1, \dots, Y_n$. The ratio N$/n$ is a natural estimator of $\bar{F}(u)$. Assume we have computed estimators $\hat{\sigma}, \hat{\gamma}$ of the parameters in the GP distribution $\bar{F}_u(x) = H(x)$. Since $\bar{F}(x) = \bar{F}(u)\bar{F}_u(x - u)$, a natural estimator of the "tail function" $\bar{F}(x)$, for $x > u$, then is

$$\hat{\bar{F}}(x) = \frac{N}{n} \, \hat{\bar{H}}(x - u) = \frac{N}{n}\left(1 + \frac{\hat{\gamma}}{\hat{\sigma}}(x - u)\right)^{-1/\hat{\gamma}}$$

Solving $1 - \hat{\bar{F}}(x_p) = p$ for x_p we get an estimator of the p-th quantile of X:

$$x_p = u + \frac{\hat{\sigma}}{\hat{\gamma}}\left(\left(\frac{n}{N}(1 - p)\right)^{-\hat{\gamma}} - 1\right), \qquad \text{for } p > 1 - \frac{N}{n}$$

(Why not just estimate $\bar{F}(x)$ by $N(x)/n$? Because if x is a very high level then $N(x)$ is small or zero, and then this estimator is useless -- and it is such very large x-es we are interested in.)
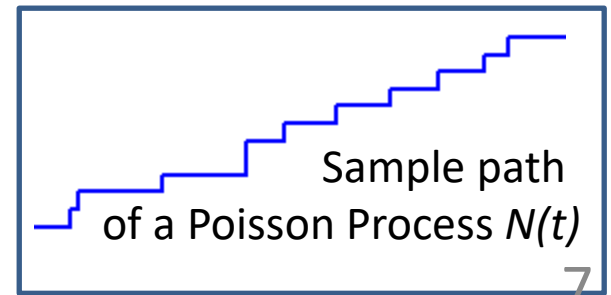
The Poisson process

Model for times of occurrence of events which occur "randomly" in time, with a constant "intensity", e.g, radioactive decay, times when calls arrive to a telephone exchange, times when traffic accidents occur ... Can be mathematically described as a counting process *N(t) = #events in [0, t].* The counting process *N(t)* is a Poisson process if

a) The numbers of events which occur in disjoint time intervals are mutually independent

b) *N(s+t) − N(s)* has a Poisson distribution for any *s, t ≥ 0, i.e.*

$$P\big(N(s+t) - N(s) = k\big) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad \text{for any} \ \ s, t \geq 0, \ k = 1, 2, \ldots$$

λ is the "intensity" parameter. It is the expected number of events in any time interval of length *1.*

Sample path
of a Poisson Process *N(t)*

A connection between the PoT and Block Maxima methods

Suppose the PoT model holds, so values larger than u occur as a Poisson process with intensity λ; this process is independent of the sizes of the excesses; these are i.i.d. and have a GP distribution $H(x) = 1 - \left(1 + \frac{\gamma}{\sigma}x\right)_+^{-1/\gamma}$. $M_T =$ the maximum in the time interval $[0, T]$. Then if $x > 0$

$$
\begin{aligned}
P(M_T \le u + x) &= \sum_{k=0}^{\infty} P(M_T \le u + x, \text{ there are k exceedances in } [0, \text{T}]) \\
&= \sum_{k=0}^{\infty} H(x)^k \frac{(\lambda T)^k}{k!} \exp\{-\lambda T\} \\
&= \sum_{k=0}^{\infty} (1 - (1 + \frac{\gamma}{\sigma}x)_+^{-1/\gamma})^k \frac{(\lambda T)^k}{k!} \exp\{-\lambda T\} \\
&= \exp\{(1 - (1 + \frac{\gamma}{\sigma}x)_+^{-1/\gamma})\lambda T\} \exp\{-\lambda T\} \\
&= \exp\{-(1 + \frac{\gamma}{\sigma}x)_+^{-1/\gamma}\lambda T\} \\
&= \exp\{-(1 + \gamma \frac{x - ((\lambda T)^\gamma - 1)\sigma/\gamma}{\sigma(\lambda T)^\gamma})_+^{-1/\gamma}\}
\end{aligned}
$$

Dependence

Often excesses of u are not independent, and then the formula on the previous slide is not valid -- however it still works if one declusters, and instead of excesses uses cluster maxima. Declustering is discussed later in the course, in Lecture 5, p- 11 - 12