Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MMG621 NONLINEAR OPTIMISATION

Date: 16–08–25
Examiner: Michael Patriksson

Question 1

(the simplex method)

(2p) a) We first rewrite the problem on standard form. We introduce slack variables s_1 and s_2. Consider the following linear program:

$$\begin{aligned} \text{minimize} \quad & z = -3x_1 - 5x_2 \\ \text{subject to} \quad & \phantom{3x_1 +{}} 2x_2 + s_1 \phantom{{}+s_2} = 12, \\ & 3x_1 + 2x_2 \phantom{{}+s_1} + s_2 = 18, \\ & x_1, \quad x_2, \quad s_1, \quad s_2 \geq 0. \end{aligned}$$

We start directly with phase II at the origin. The starting basis is $(s_1, s_2)^{\mathrm{T}}$. Calculating the reduced costs for the non-basic variables x_1, x_2 we obtain $\tilde{\boldsymbol{c}}_N = (-3, -5)^{\mathrm{T}}$, meaning that x_2 enters the basis. From the minimum ratio test, we get that s_1 leaves the basis.

Updating the basis we now have $(x_2, s_2)^{\mathrm{T}}$ in the basis. Calculating the reduced costs, we obtain $\tilde{\boldsymbol{c}}_N = (-12, 5/2)^{\mathrm{T}}$, meaning that x_1 enters the basis. From the minimum ratio test we get that s_2 leaves the basis.

Updating the basis we now have $(x_1, x_2)^{\mathrm{T}}$ in the basis. Calculating the reduced costs, we obtain $\tilde{\boldsymbol{c}}_N = (3/2, 2)^{\mathrm{T}}$, meaning that the current basis is optimal. The optimal solution is thus

$$\boldsymbol{x}^* = (x_1, x_2, s_1, s_2)^{\mathrm{T}} = (2, 6, 0, 0)^{\mathrm{T}},$$

with optimal objective value $f^* = 36$.

(1p) b) Since there is an optimal solution to the problem, Strong duality guarantees the existence of a dual optimal solution. The dual optimal solution is $\boldsymbol{y}^{*\mathrm{T}} = \boldsymbol{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} = (-3/2, -1)$. The optimal basis is not degenerate. The optimal solution is thus unique.

Question 2

(the KKT conditions)

(1p) a) See the Book, system (5.9).

(1p) b) The vector $\boldsymbol{x}^1$ satisfies the KKT conditions (5.9).

(1p) c) Nothing. (Under the conditions given, there may be optimal solutions that do not satisfy the KKT conditions.)

(3p) Question 3

(Lagrangian duality)

The Lagrange function is

$$\begin{aligned} L(x,\mu) &= -x_1 - 2x_2 + \mu_1(x_1^2 + x_2^2 - 1) + \mu_2(x_1 + 0.5x_2 - 1) \\ &= \underbrace{\mu_1 x_1^2 + (\mu_2 - 1)x_1}_{q_1(x_1)} + \underbrace{\mu_1 x_2^2 + (0.5\mu_2 - 2)x_2}_{q_2(x_2)} - \mu_1 - \mu_2. \end{aligned}$$

When $\mu_1 < 0$, $L(x,\mu)$ is strictly concave with respect to x which makes $\min_{x_1} q_1(x_1)$ and $\min_{x_2} q_2(x_2)$ unbounded from below. Similarly, when $\mu_1 = 0$, $L(x,\mu)$ is linear and at least one of $\min_{x_1} q_1(x_1)$ and $\min_{x_2} q_2(x_2)$ is unbounded from below. Only when $\mu_1 > 0$ is $L(x,\mu)$ strictly convex with respect to x, and $\min_x L(x,\mu)$ is finite. In this case, the minimizers of q_1 and q_2 are, respectively,

$$x_1(\mu) = \frac{1-\mu_2}{2\mu_1}, \quad x_2(\mu) = \frac{2-0.5\mu_2}{2\mu_1}. \tag{1}$$

Consequently, the dual function is

$$q(\mu) = \begin{cases} -\frac{1}{4\mu_1}\left((1-\mu_2)^2 + (2-0.5\mu_2)^2\right) - \mu_1 - \mu_2, & \text{when } \mu_1 > 0 \\ -\infty, & \text{when } \mu_1 \le 0 \end{cases}.$$

The dual problem is

$$\begin{aligned} \text{minimize} \quad & q(\mu) \\ \text{subject to} \quad & \mu_1 > 0 \end{aligned}.$$

The dual function q is differentiable as expressed. The dual problem is always convex.

Since the primal problem is convex and the Slater constraint qualifications hold, strong duality holds. Hence, the duality gap is zero and the optimal dual solution is attained (which is the same as the Lagrangian multiplier).

There are multiple ways to obtain the optimal primal and dual solutions. An approach is as follows: By graphically inspecting the primal problem, it can be seen that $(1,0)^T$ is the optimal primal solution. Then, by Theorem 6.9 in the text, if $x^\star = (1,0)^T$ and $\mu^\star = (\mu_1^\star, \mu_2^\star)^T$ are the optimal primal and dual pair, they must satisfy (1). This implies that $\mu_1^\star = -\frac{3}{2}$ and $\mu_2^\star = 4$.

(3p) Question 4

(modelling)

Introduce the binary variables

$$x_i = \begin{cases} 1 & \text{if team } i \text{ is in group 1} \\ 0 & \text{otherwise} \end{cases}, \qquad i = 1, \ldots, 14.$$

The objective is to minimize the function

$$\sum_{i=1}^{13} \sum_{j=i+1}^{14} d_{ij}(x_i x_j + (1 - x_i)(1 - x_j)).$$

The constraints are

$$\begin{aligned}
\sum_{i=1}^{14} x_i &= 7 \\
x_1 + x_2 &= 1 \\
\sum_{i=1}^{14} x_i p_i &= \sum_{i=1}^{14} (1 - x_i) p_i + 0.2 \sum_{i=1}^{14} p_i \\
\sum_{i=1}^{14} (1 - x_i) p_i &= \sum_{i=1}^{14} x_i p_i + 0.2 \sum_{i=1}^{14} p_i \\
x_i &\in \{0, 1\}, \qquad i = 1, \ldots, 14
\end{aligned}$$

The first constraint makes sure that there are 7 teams in each group. The second constraint ensures that the two best teams are not in the same group. The third and the fourth constraints ensure that the groups are arranged so that the difference between the sum of points in the two groups are not bigger than 20% of the total points.

Question 5

(true or false)

(1p) a) False – f may be discontinuous, for example.

(1p) b) False – there may be *no* rounding that is even feasible.

(1p) c) True – the linear program describing the Phase I problem is a linear program with an objective function that is bounded from below by zero. Since the objective value is bounded the extreme point with the lowest objective value is optimal.

(3p)

Question 6

(global convergence of a penalty method)

See Theorem 13.4.

Question 7

(the KKT conditions)

(2p) a) The KKT conditions are

$$\begin{aligned}
\nabla f(\mathbf{x}) + \lambda \nabla h(\mathbf{x}) + \mu \nabla g(\mathbf{x}) = \begin{pmatrix} 2x_1 + \lambda \\ 2x_2 + \lambda \\ 2x_3 + \lambda \\ 2x_4 + \lambda + \mu \end{pmatrix} &= \mathbf{0}, && (1) \\
x_1 + x_2 + x_3 + x_4 &= 1, && (2) \\
x_4 &\leq A, && (3) \\
\mu &\geq 0, && (4) \\
\mu(x_4 - A) &= 0, && (5)
\end{aligned}$$

giving that $x_1 = x_2 = x_3 = -\lambda/2$ and $x_4 = (-\lambda - \mu)/2$. From (1) we then get that $\lambda = (-2 - \mu)/4$ and thus $x_1 = x_2 = x_3 = 1/4 + \mu/8$ and $x_4 = 1/4 - 3\mu/8$.

From (2) we get that $3\mu/8 \geq 1/4 - A$; we treat the following three cases individually.

1. Assume that $A > 1/4$, implying that $\mu \geq 0$, $x_1 = x_2 = x_3 \geq 1/4$ and $x_4 = 1 - (x_1 + x_2 + x_3) \leq 1/4$. From (4) it follows that $\mu = 0$ and the optimal solution hence is $x_1 = x_2 = x_3 = x_4 = 1/4$.
2. $A = 1/4$ leads to the same optimal solution as the case above.
3. Assume that $A \leq 1/4$. Let $x_4 < A$; then $\mu = 0$ and $x_4 = 1/4 > A$. Therefore, $x_4 = A$ and $x_1 = x_2 = x_3 = 1/3(1 - A)$. Then, the

original problem reduces to the minimization of $1/3(1-A^2)+A^2 = 1/3(1-2A+4A^2)$ which is always $\geq 1/4$ and $1/3(1-A^2)+A^2 = 1/4$ for $A = 1/4$. The optimal solution is thus $x_1 = x_2 = x_3 = x_4 = 1/4$.

(1p) b) The objective function of the problem considered can be written as a function of the parameter A as

$$f(A) = \begin{cases} \frac{1}{4} & \text{if } A \geq 1/4, \\ \frac{1}{3}(1-2A+4A^2) & \text{otherwise.} \end{cases}$$