1. Introduction.

You have learned how to solve

$$\Delta u = 0 \quad \text{in } D$$
$$u = f \quad \text{on } \partial D$$

when D is $\mathbb{R}^n_+$ or a ball $B_1(0)$.

This is an incredable result - but often one would like to solve the equation in more general domains.

When D is a halfspace or $B_1(0)$ then the domain has much symmetry which allows us to explicitly construct a solution formula.

For instance in $\mathbb{R}^n_+$

$$u(x) = \frac{\pm 1}{\cancel{\omega_n}} \int_{\mathbb{R}^{n-1}} \frac{x_n}{|x-y|^n} f(y')\,dy' = \int_{\mathbb{R}^{n-1}} \frac{\partial N(x,y)}{\partial \nu_y} f(y')\,dy'$$

$$N = \frac{-1}{(n-2)\omega_n} \frac{1}{|x-y|^{n-2}} \qquad \nu_y = -e_n.$$

We might guess that a solution to the Dirichlet problem in D is given by

$$u(x) = \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} f(y)\,d\sigma(y) \qquad (1)$$

area measure on ∂D

And that

$$u(x) = \int_{\partial D} N(x,y) f(y)\,d\sigma(y)$$ would solve the Neumann problem.

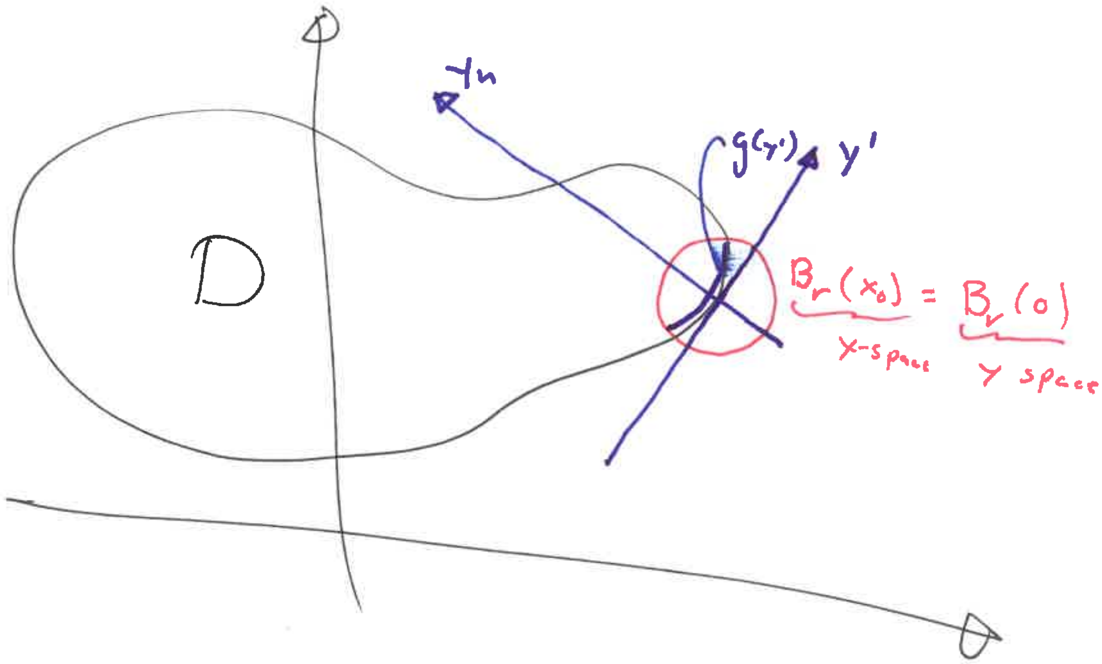
$$\Delta u = 0 \quad \text{in } D$$
$$\frac{\partial u}{\partial \nu} = f \quad \text{on } \partial D$$

This will not exactly work - but it will give us a hint of a way to prove existence.

First we will have to define the domains that we will be able to work in.

Definition: We say that $D \subset \mathbb{R}^n$ is a $C^{1,\alpha}$ domain if there exist an $r > 0$ such that for every $x_0 \in \partial D$ there is a coordinate system s.t. $\partial D \cap B_r(x_0) = \{(y', g(y')) ;\ x' \in B_r'(0)\}$ for some function $g(x')$ that is $C^{1,\alpha}$; that is $g(x)$ is continuously differentiable and

$$\|g\|_{C^{1,\alpha}} = \|g\|_{C^0} + \|\nabla g\|_{C^0} + \sup_{x,y} \frac{|\nabla g(x) - \nabla g(y)|}{|x-y|^\alpha} \leq C_0$$



Aim for today: If u is defined by (1)

1. ~~what~~ will $\Delta u = 0$ in D

2. what is $\lim\limits_{\substack{x \to x_0 \in \partial D \\ x \in D}} u(x)$? (hopefully $f(x_0)$)

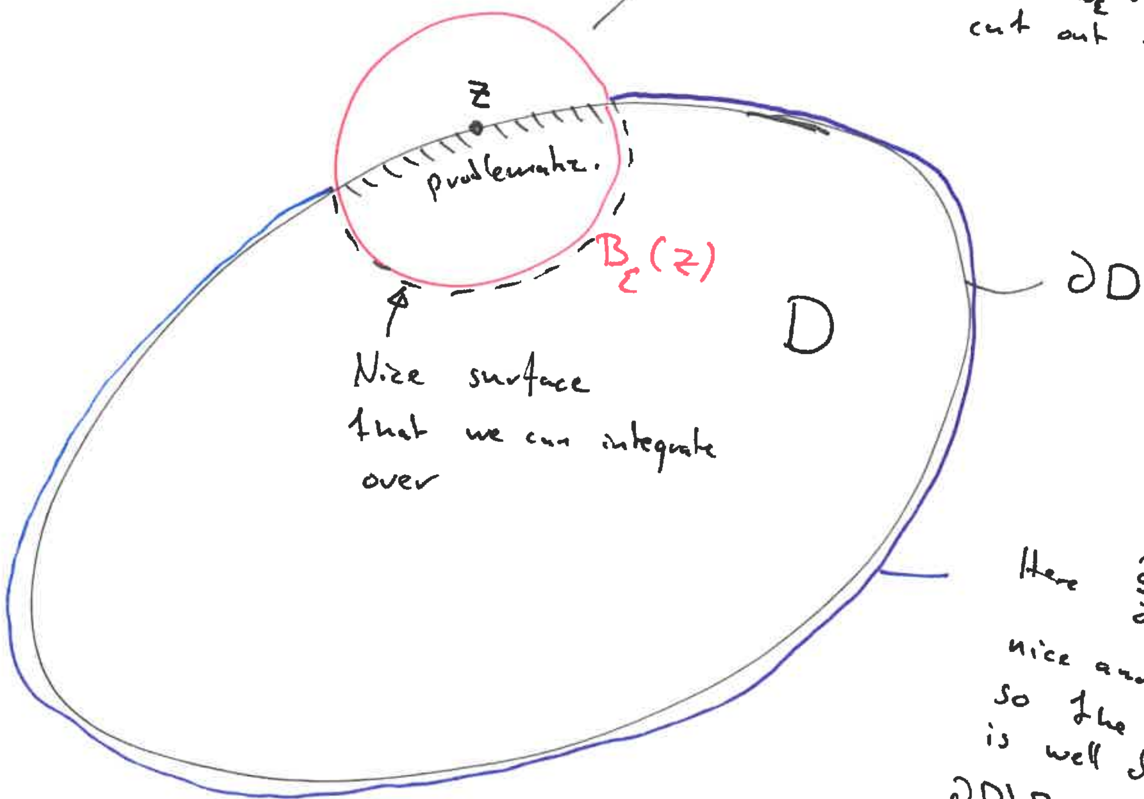
<u>Proposition:</u> The function u defined by (2) is harmonic in D.

<u>Proof:</u> Differentiate under the integral sign and use that $\Delta_x N(x,y) = 0$. $\square$

In order to ~~prove~~ calculate the limit in 2.:

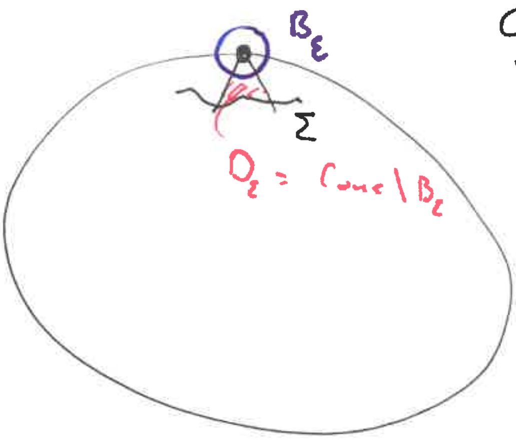
$$\lim_{\substack{x \to z \in \partial D \\ x \in D}} \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} f(y)\, d\sigma(y).$$

1) $\frac{\partial N}{\partial \nu_y}$ has a bad singularity at z. Cut out a small ball $B_\varepsilon(z)$ to cut out the singularity



Here $\frac{\partial N}{\partial \nu_y}$ is nice and smooth so the integral is well defined over $\partial D \setminus B_\varepsilon(z)$

Lemma: Let Σ be a piece of C^1 surface, with C^1 boundary, not intersecting the origin. We also assume that each ray through the origin only intersect Σ in one point.

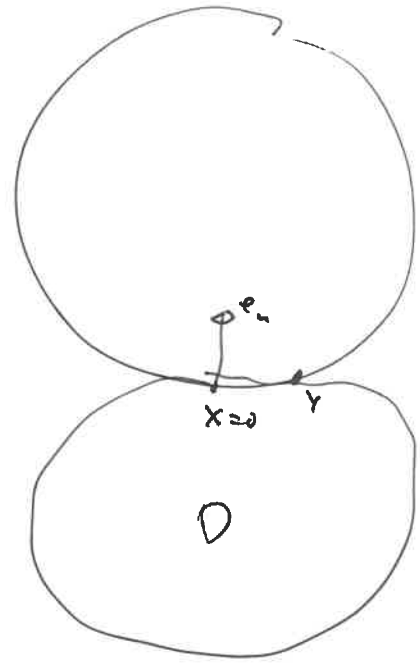


Then
$$\int_\Sigma \frac{\partial N(0,y)}{\partial \nu_y}\, d\sigma(y) = \frac{\alpha}{\omega_n},$$
where α is the solid angle of the cone of rays from the origin through the surface.

Proof: $N(0,y)$ is harmonic so
$$0 = \int_{D_\varepsilon} \Delta_y (N(0,y))\, dy = \underbrace{\int_\Sigma \frac{\partial N(0,y)}{\partial \nu_y}\, d\sigma(y)} + \underbrace{\int_{\partial B_\varepsilon(0)\cap D} \frac{\partial N(0,y)}{\partial \nu_y}\, d\sigma(y)}_{\int_{\partial B_\varepsilon \cap D} -\frac{1}{(n-2)\omega_n} \frac{\nu_y \cdot (x-y)}{|x-y|^n}\, d\sigma} + \underbrace{\int_{\partial D \setminus (\Sigma \cup B_\varepsilon)} \frac{\partial N(0,y)}{\partial \nu_y}\, d\sigma(y)}_{=0 \text{ since } \frac{\partial N}{\partial \nu_y} = 0}.$$

Lemma: Given a bounded $C^{1,\alpha}$-domain D, there is a constants C_D, that depend on the dimension and on the domain D, ~~such that for any $x_0 \in \partial D$ and $r < r_0$ and~~ $x, y \in B_r(x_0) \cap D$
$$\left| \frac{\partial N(x,y)}{\partial \nu_y} \right| = \left| \frac{1}{(n-2)\omega_n} \nu_y \cdot \nabla_y \frac{1}{|x-y|^{n-2}} \right| \le \frac{C_D}{|x-y|^{n-1-\alpha}}$$

Proof: We may assume that $0 = x$ and that $\nu(x) = e_n$. ∂D is a $C^{1,\alpha}$ domain so it is given by the graph of a $C^{1,\alpha}$ function f. So



$$\nu(y) = \frac{(-\nabla' f(y'), 1)}{\sqrt{1+|\nabla' f|^2}}$$

So

$$|\nu(y)\cdot\nabla N(0,y)| = \left|\frac{1}{\omega_n}\,\frac{\nu_y\cdot(\cancel{x}^{\,0}-y)}{|\cancel{x}^{\,0}-y|^n}\right| =$$

$$= \left|\frac{1}{\omega_n}\,\frac{(-\nabla' f, 1)\cdot(y', f(y'))}{\sqrt{1+|\nabla' f|^2}\;|y|^n}\right| \leq$$

$$\leq C\,\frac{|y'|\overbrace{|\nabla' f|}^{C|y'|^\alpha} + \underbrace{|f(y')|}_{C|y'|^{1+\alpha}}}{|y|^n} \leq C\,\frac{1}{|y|^{n-\alpha-1}}.$$

$\blacksquare$

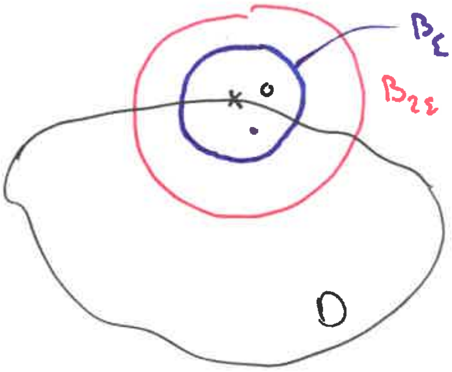
Lemma: Given a bounded $C^{1,\alpha}$-domain D there exists constants $C_1, c_0 > 0$ depending on D such that for every $x_0 \in \partial D$ and every $r < c_0$ and $x \in B_r(x_0) \cap D$

$$\int_{B_r(x_0)\cap \partial D} \left| \frac{(x-y)\cdot \nu_y}{|x-y|^n} \right| d\sigma(y) \leq C_1 .$$

B estimate from above

A estimate from below

Proof: We may assume $x_0 = 0$ and ε is so small that $D \cap B_{2\varepsilon}(0)$ is a graph of a $C^{1,\alpha}$ function also pick $x \in B_\varepsilon(0)$. Then $\exists \bar{x} \in \partial D \cap B_{2\varepsilon}$ s.t. x laies on the normal line of $\bar{x}$. Set $\delta = |x - \bar{x}|$.

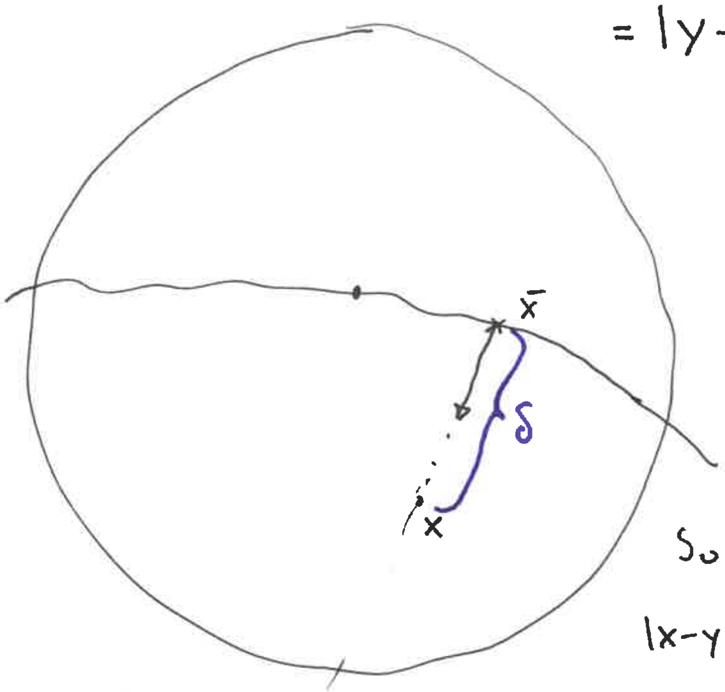


(A) Then

$$|x-y|^2 = |(x-\bar{x}) - (y-\bar{x})|^2 =$$

$$= \underbrace{|y-\bar{x}|^2 - 2\underbrace{(\bar{x}-x)}_{=\delta\nu_{\bar{x}}}\cdot(\bar{x}-y)}_{\leq\ \delta \nu_{\bar{x}}\cdot(\bar{x}-y)} + |x-\bar{x}|^2$$

$$\delta\underbrace{\nu_{\bar{x}}\cdot(\bar{x}-y)}_{\leq C|\bar{x}-y|^{1+\alpha}}$$



So

$$|x-y|^2 \geq |\bar{x}-y|^2 + \delta^2 - C\delta|\bar{x}-y|^{1+\alpha} \geq$$

$$\geq \frac{1}{4}\left(|\bar{x}-y|^2 + \delta^2\right)$$

if $|\bar{x}-y| < \varepsilon$ is small enough

B) To estimate the numerator we use the triangle inequality

$$\left|(x-y)\cdot \nu_y\right| \leq \underbrace{\left|(\bar{x}-y)\cdot \nu_y\right|} + \left|\underbrace{(x-\bar{x})}_{\text{length } \delta}\cdot \nu_y\right| \leq C\,|\bar{x}-y|^{1+\alpha} + \delta^2.$$

Therefore

$$\frac{|(x-y)\cdot\nu_y|}{|x-y|^n} \leq C\,\frac{|\bar{x}-y|^{1+\alpha}+\delta^2}{\left(|\bar{x}-y|+\delta\right)^n} \qquad \text{so}$$

$$\int_{B_\varepsilon\cap\partial D}\left|\frac{(x-y)\cdot\nu_y}{|x-y|^n}\right| d\sigma(y) \leq \underbrace{\int_{B_\varepsilon\cap\partial D}\frac{C}{|\bar{x}-y|^{n-1-\alpha}}\,d\sigma}_{\leq C\varepsilon^\alpha} + C\delta^2\underbrace{\int_{B_\varepsilon\cap\partial D}\frac{1}{(|\bar{x}-y|+\delta)^n}\,d\sigma}_{\leq C}$$

☒

We can now calculate the boundary data of

$$u(x) = \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} f(y)\, d\sigma \qquad (1)$$

Thm: If $f \in C(\partial D)$ then we may extend $u(x)$ to a continuous function on $\overline{D}$ and

$$u(x) = + \frac{1}{2} f(x) + \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} f(y)\, d\sigma(y) \qquad (2)$$

(similar for Neumann).

Proof: We can split the integral into $\int_{\partial D} = \int_{\partial D \cap B_\varepsilon} + \int_{\partial D \setminus B_\varepsilon}$

We want to show that for $z \in D$ $z \to x \in \partial D$ ~~and~~ then $u(z)$ converges to the right side in (2)

Note that

$$\int_{\partial D} \frac{\partial N(z,y)}{\partial \nu_y} f(y)\, d\sigma(y) = \overbrace{f(x) \int_{\partial D \cap B_\varepsilon(x)} \frac{(z-y)\cdot \nu_y}{\omega_n |z-y|^n}\, d\sigma(y)}^{I_1} +$$

cancel

$$+ \underbrace{\int_{\partial D \cap B_\varepsilon} (f(y) - f(x)) \frac{(z-y)\cdot \nu_y}{\omega_n |z-y|^n}\, d\sigma(y)}_{I_2} + \underbrace{\int_{\partial D \setminus B_\varepsilon(x)} f(y) \frac{(z-y)\cdot \nu_y}{\omega_n |x-y|^n}\, d\sigma(y)}_{I_3}$$

To estimate I_1 we use Lemma 1

$$I_1 = + f(x) \frac{\alpha(z,x,\varepsilon)}{\omega_n} \longrightarrow + \frac{1}{2} f(x)$$

solid angle

as $z \to x$ & $\varepsilon \to 0$

$$|I_2| \le \underbrace{\sup_{y \in B_\varepsilon(x) \cap \partial D} \left(|f(y) - f(x)| \right)}_{\substack{\to 0 \\ \text{as } \varepsilon \to 0}} \underbrace{\left| \int_{B_\varepsilon \cap \partial D} \frac{(z-y)\cdot \nu_y}{|z-y|^n} \, d\sigma(y) \right|}_{\text{bounded by lemma 2}} \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

$$I_3 \longrightarrow \int_{\partial D} f(y) \frac{(z-y)\cdot \nu_y}{\omega_n |z-y|^n} \, d\sigma(y).$$

Put I_1, I_2 and I_3 together gives the result...

To see this we use

$$I_1 = \underbrace{f(x) \int_{\partial D \cap B_\varepsilon(x)} \frac{(z-y)\cdot \nu_y}{\omega_n |z-y|^n} \, d\sigma(y) + f(x) \int_{\partial B_\varepsilon \cap D} \frac{(z-y)\cdot \nu_y}{\omega_n |z-y|^n} \, d\sigma(y)}_{= \underbrace{\int_{D \cap B_\varepsilon} \Delta \left(-\frac{1}{(n-2)\omega_n} \frac{1}{|z-y|^{n-2}} \right) dy}_{=1}} - f(x) \int_{\partial B_\varepsilon \cap D} \frac{(z-y)\cdot \nu_y}{\omega_n |z-y|^n} \, d\sigma$$

$$= f(x) - \underbrace{f(x) \int_{\partial B_\varepsilon \cap D} \frac{(z-y)\cdot \nu_y}{\omega_n |z-y|^n} \, d\sigma(y)}_{= f(x) \underbrace{\frac{\alpha(z,x,\varepsilon)}{\omega_n}}_{\to \frac{1}{2}}}$$

Strategy.

If we want to solve

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases}$$

it is enough to find a function $\varphi(x)$ s.t.

$$f(x) = \frac{1}{2}\varphi(x) + \underbrace{\int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} \varphi(y)\, d\sigma(y)}_{T(\varphi)}.$$

Thus we can define the operator T and we need to show that

$\frac{1}{2} + T$ is an operator that is onto.

Therefore we need

1) To define domains of definition of T
   Hilbert & Banach spaces L2
2) Investigate properties of mappings from spaces to spaces
   Riesz - Schauder (Fredholm Theory) L3 - L4
3) Show existence L5.
4) We have two more lectures, don't know what to do...