FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Hilbert spaces

Why should we bother to understand Hilbert spaces? Hilbert spaces are important because they are the missing mathematics (which Fourier did not have!) to rigorously justify using Fourier series to solve PDEs. We have learned the following procedure:

(1) Start with a PDE where the x variable is in a finite (bounded) interval.
(2) Separate variables by writing u, (the unsub) as a product like $u(x,t) = X(x)T(t)$. Plug it into the PDE.
(3) Solve for X using the boundary conditions. This will probably give lots of Xs which can be indexed by $\mathbb{N}$.
(4) Each X_n has a partner T_n. Solve for these. Probably, you've got some unknown constants.
(5) Is the PDE homogeneous? If so, $X_1T_1 + X_2T_2 + \dots$ also solves the PDE so you can smash them together into a big party series. If *not* then you may need to do something else (i.e. steady state solution). In the homogeneous case, you will then use the IC and the collection $\{X_n\}$ to find the coefficients in T_n and end up with a solution of the form
$$\sum_{n\in\mathbb{N}} X_n(x)T_n(t).$$
It's precisely in this last step where the Hilbert space theory is being used to say that you can use the X_n obtain the IC, because the Hilbert space theory tells us when certain functions are basis functions for $\mathcal{L}^2$!

A *Hilbert space* is a complete[1], normed vector space whose norm is defined by a scalar product. The definition of a vector space means that if u and v are elements in your Hilbert space, then for all complex numbers a and b,
$$au + bv \text{ is in your Hilbert space.}$$
So, taking $a = b = 0$, there is always a 0 vector in your Hilbert space. The fact that it is normed means that every element of the Hilbert space has a *length,* which

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[1] Every Cauchy sequence converges. Do you remember what a Cauchy sequence is? If not, please look it up or ask!

is equal to its norm. To define this, we describe the scalar product. For a Hilbert space H, the scalar product satisfies:

$$u, v \in H \implies \langle u, v \rangle \in \mathbb{C},$$
$$c \in \mathbb{C} \implies \langle cu, v \rangle = c \langle u, v \rangle,$$
$$u, v, w \in H \implies \langle u + w, v \rangle = \langle u, v \rangle + \langle w, v \rangle,$$
$$\langle u, v \rangle = \overline{\langle v, u \rangle},$$
$$\langle u, u \rangle \geq 0, \quad = 0 \iff u = 0.$$

Therefore, we can define the norm of a vector as

$$||u|| := \sqrt{\langle u, u \rangle}.$$

The norm of a vector is also equal to its distance from the 0 element of the Hilbert space. Similarly,

$$||u - v|| = \sqrt{\langle u - v, u - v \rangle}$$

is the distance between the elements u and v in your Hilbert space. We say that a set of elements

$$\{u_\alpha\} \subset H$$

is an orthonormal basis (ONB) for H if for any $v \in H$ there exist complex numbers (c_α) such that

$$v = \sum c_\alpha u_\alpha, \quad \langle u_\alpha, u_\beta \rangle = \delta_{\alpha,\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta. \end{cases}$$

This is the Kronecker δ. You may be wondering why we haven't written an index for α. Well, that's because à priori, they could be uncountable.

Theorem 1. *A Hilbert space is separable if and only if it has either a finite ONB or a countable ONB.*

There is a cute proof here:
`http://www.polishedproofs.com/relationship-between-a-countable-orthonormal-basis-and-a-cou`

We're only going to be working with Hilbert spaces which have either a finite ONB or a countable ONB. The dimension of a Hilbert space is the number of elements in an ONB. Any finite dimensional Hilbert space is in bijection with the standard one

$$\mathbb{C}^n, \quad u, v \in \mathbb{C}^n \implies \langle u, v \rangle = u \cdot \overline{v}.$$

Thus, writing

$$u = (u_1, \ldots, u_n), \qquad \text{with each component } u_k \in \mathbb{C},\ k = 1, \ldots, n$$

and similarly for v,

$$\langle u, v \rangle = \sum_{k=1}^{n} u_k \overline{v_k}.$$

The bijection between any finite (n) dimensional Hilbert space and $\mathbb{C}^n$ comes from taking an ONB of the Hilbert space and mapping the elements of the ONB to the standard basis vectors of $\mathbb{C}^n$. Here are some useful basic results for Hilbert spaces.

Proposition 2. *Let H be a Hilbert space. For any u and v in H,*

$$||u + v||^2 = ||u||^2 + 2\Re\langle u, v \rangle + ||v||^2.$$

Proof: Compute:

$$\begin{aligned}||u+v||^2 &= \langle u+v, u+v\rangle = \langle u, u+v\rangle + \langle v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,v\rangle + \langle v,u\rangle \\ &= ||u||^2 + \langle u,v\rangle + ||v||^2 + \overline{\langle u,v\rangle}.\end{aligned}$$

We all know that for a complex number z,

$$z + \overline{z} = 2\Re(z).$$

So,

$$\langle u,v\rangle + \overline{\langle u,v\rangle} = 2\Re\langle u,v\rangle.$$

1.1. **Cauchy-Schwarz Inequality, Triangle Inequality, and Pythagorean Theorem.**

Proposition 3. *For any Hilbert space, H, for any u and v in H,*

$$|\langle u,v\rangle| \le ||u||||v||.$$

Proof: Assume that at least one of the two is non-zero. Let's assume $v \neq 0$, because otherwise we can just swap their names. We begin by considering the length of the vector u plus v scaled by a factor of t. If $t \to 0$, the length tends to $||u||^2$. What happens for other values of t? We compute it:

$$||u+tv||^2 = ||u||^2 + 2t\Re\langle u,v\rangle + t^2||v||^2, \quad t \in \mathbb{R}.$$

This is a real valued function of t. It's a quadratic function of t in fact. The derivative is

$$2t||v||^2 + 2\Re\langle u,v\rangle.$$

It's an upwards shaped quadratic function, so its unique minimum is when

$$t = -\frac{\Re\langle u,v\rangle}{||v||^2}.$$

If we then check out what happens at this value of t,

$$||u+tv||^2 = ||u||^2 - 2\frac{\Re\langle u,v\rangle}{||v||^2}\Re\langle u,v\rangle + \Re\langle u,v\rangle^2\frac{||v||^2}{||v||^4} = ||u||^2 - \frac{\Re\langle u,v\rangle^2}{||v||^2}.$$

We know that

$$0 \le ||u+tv||^2$$

so we get

$$0 \le ||u||^2 - \frac{\Re\langle u,v\rangle^2}{||v||^2} \implies 0 \le ||u||^2||v||^2 - \Re\langle u,v\rangle^2.$$

This gives us

$$\Re\langle u,v\rangle^2 \le ||u||^2||v||^2.$$

Well, this is annoying because of that silly $\Re$. I wonder how we could make it turn into $|\langle u,v\rangle|$? Also, we don't want to screw up the $||u||^2||v||^2$ part. Well, we know how the scalar product interacts with complex numbers, for $\lambda \in \mathbb{C}$,

$$\langle \lambda u, v\rangle = \lambda\langle u,v\rangle.$$

So, if for example

$$\langle u,v\rangle = re^{i\theta}, r = |\langle u,v\rangle| \text{ and } \theta \in \mathbb{R}.$$

We can modify u, without changing $||u||$,
$$||e^{-i\theta}u|| = ||u||.$$
Moreover
$$\langle e^{-i\theta}u, v\rangle = e^{-i\theta}\langle u, v\rangle = e^{-i\theta}re^{i\theta} = |\langle u, v\rangle|.$$
So, if we repeat everything above replacing u with $e^{-i\theta}u$ we get
$$\Re\langle e^{-i\theta}u, v\rangle^2 \leq ||e^{-i\theta}u||^2||v||^2 = ||u||^2||v||^2,$$
and by the above calculation
$$\langle e^{-i\theta}u, v\rangle = |\langle u, v\rangle| \in \mathbb{R} \implies \Re\langle e^{-i\theta}u, v\rangle^2 = |\langle u, v\rangle|^2.$$
So, we have
$$|\langle u, v\rangle|^2 \leq ||u||^2||v||^2.$$
Taking the square root of both sides completes the proof of the Cauchy-Schwarz inequality.

We also have a triangle inequality.

Proposition 4. *For any u and v in a Hilbert space H,*
$$||u + v|| \leq ||u|| + ||v||.$$

Proof: We just use the previous two results:
$$||u+v||^2 = ||u||^2 + 2\Re\langle u, v\rangle + ||v||^2 \leq ||u||^2 + 2||u||||v|| + ||v||^2 = (||u|| + ||v||)^2.$$
Taking the square root we obtain the triangle inequality.

We have the Pythagorean theorem.

Proposition 5. *If u and v are orthogonal, then*
$$||u + v||^2 = ||u||^2 + ||v||^2.$$
Moreover, if $\{u_n\}_{n=1}^N$ are orthogonal, then
$$||\sum_{n=1}^N u_n||^2 = \sum_{n=1}^N ||u_n||^2.$$

Proof: The first statement follows from
$$||u+v||^2 = ||u||^2 + 2\Re\langle u, v\rangle + ||v||^2 = ||u||^2 + ||v||^2,$$
if u and v are orthogonal, because in that case their scalar product is zero. Moreover, for any collection of orthogonal vectors $\{u_1, \ldots, u_n\}$ we proceed by induction. Assume that
$$||u_1 + \ldots + u_{n-1}||^2 = \sum_{k=1}^{n-1} ||u_k||^2.$$
Then, if u_n is orthogonal to all of $u_1, \ldots, u_{n-1}$ we also have
$$\langle u_n, u_1 + \ldots + u_{n-1}\rangle = \langle u_n, u_1\rangle + \ldots + \langle u_n, u_{n-1}\rangle = 0 + \ldots + 0.$$

Hence u_n is also orthogonal to the sum,

$$\sum_{k=1}^{n-1} u_k.$$

By the Pythagorean theorem,

$$||u_n + \sum_{k=1}^{n-1} u_k||^2 = ||u_n||^2 + ||\sum_{k=1}^{n-1} u_k||^2.$$

By the induction assumption

$$= ||u_n||^2 + \sum_{k=1}^{n-1} ||u_k||^2 = \sum_{k=1}^{n} ||u_k||^2.$$

1.2. **Continuity of the scalar product.**

Proposition 6. *Using only the assumptions that the scalar product satisfies:*

$$\langle u, v\rangle = \overline{\langle v, u\rangle}$$
$$\langle au, v\rangle = a\langle u, v\rangle$$
$$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$$
$$\langle u, u\rangle \geq 0, \quad \langle u, u\rangle = 0 \iff u = 0,$$

then the scalar product is a continuous function from $H \times H \to \mathbb{C}$.

Proof: It suffices to estimate

$$|\langle u, v\rangle - \langle u', v'\rangle| .$$

I would like to somehow get

$$u - u' \text{ and } v - v'.$$

So, well, just throw them in the first and last

$$\langle u - u', v\rangle = \langle u, v\rangle - \langle u', v\rangle.$$

That shows that

$$\langle u - u', v\rangle + \langle u', v\rangle = \langle u, v\rangle.$$

So, we see that

$$\langle u, v\rangle - \langle u', v'\rangle = \langle u - u', v\rangle + \langle u', v\rangle - \langle u', v'\rangle$$

We can smash the last two terms together because $-1 \in \mathbb{R}$ so

$$-\langle u', v'\rangle = \langle u', -v'\rangle \implies \langle u', v\rangle - \langle u', v'\rangle = \langle u', v - v'\rangle.$$

Hence,

$$|\langle u, v\rangle - \langle u', v'\rangle| = |\langle u - u', v\rangle + \langle u', v - v'\rangle| .$$

By the triangle inequality

$$|\langle u - u', v\rangle + \langle u', v - v'\rangle| \leq |\langle u - u', v\rangle| + |\langle u', v - v'\rangle| .$$

By the Cauchy-Schwarz inequality

$$|\langle u - u', v\rangle| + |\langle u', v - v'\rangle| \leq ||u - u'||||v|| + ||u'||||v - v'||.$$

We therefore see that for any fixed pair $(u, v) \in H \times H$, given $\epsilon > 0$, we can define

$$\delta := \min\left\{\frac{\varepsilon}{2(||v||+1)}, \frac{\varepsilon}{2(||u||+1)}, 1\right\}.$$

Then we estimate

$$||u - u'|| < \delta \implies ||u'|| < ||u|| + \delta \leq ||u|| + 1,$$

$$||u - u'||||v|| \leq \frac{\varepsilon||v||}{2(||v||+1)} < \frac{\varepsilon}{2}.$$

and

$$||u'||||v - v'|| \leq \frac{(||u||+1)\varepsilon}{2(||u||+1)} \leq \frac{\varepsilon}{2},$$

so we obtain

$$|\langle u, v\rangle - \langle u', v'\rangle| < \varepsilon.$$

Remark 1. *This fact is useful because it allows us to bring limits inside the scalar product. You will see that we do this many times! In particular, if one has two sequences,*

$$\{u_n\}_{n\geq 1}, \quad \{v_n\}_{n\geq 1} \text{ in a Hilbert space, } H,$$

and

$$\lim_{n\to\infty} u_n = u \in H, \quad \lim_{n\to\infty} v_n = v \in H,$$

then the continuity of the scalar product implies that

$$\lim_{n\to\infty} \langle u_n, v_n\rangle = \langle u, v\rangle.$$

This fact allows us to prove an infinite dimensional Pythagorean theorem!

Theorem 7 (Infinite dimensional Pythagorus)**.** *Assume that $\{u_k\}_{k\geq 1}$ are in a Hilbert space, and that*

$$\sum_{k\geq 1} u_k$$

converges to an element u in that Hilbert space. Further, assume that the u_k are pairwise orthogonal. Then we have

$$||u||^2 = \sum_{k\geq 1} ||u_k||^2.$$

Proof: The meaning of

$$\sum_{k\geq 1} u_k = u$$

is that

$$\lim_{n\to\infty} \sum_{k=1}^{n} u_k = u.$$

This is equivalent to

$$\lim_{n\to\infty} ||u_k - u|| = 0.$$

The definition of scalar product says that

$$||u||^2 = \langle u, u\rangle.$$

Let us denote

$$U_n := \sum_{k=1}^{n} u_k.$$

Since it is a finite sum of elements of the Hilbert space, this is an element of the Hilbert space, because Hilbert spaces are vector spaces. The continuity of the scalar product shows that

$$\lim_{n\to\infty} \langle U_n, U_n \rangle = \langle U, U \rangle.$$

For each n, we also have

$$\langle U_n, U_n \rangle = \sum_{k=1}^{n} ||u_k||^2,$$

by the usual (finite) Pythagorean Theorem. Hence, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} ||u_k||^2 = ||U||^2.$$

This shows that the sum on the left converges and is equal to $||U||^2$.

1.3. Bessel's inequality and the three equivalent conditions to be an ONB.

We prove a very useful inequality.

Theorem 8 (Bessel's Inequality for general Hilbert spaces)**.** *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal set in a Hilbert space H. Then if $f \in H$,*

$$g := \sum_{n\in\mathbb{N}} \langle f, \phi_n \rangle \phi_n \in H,$$

and we have the inequality

$$||g||^2 = \sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2 \le ||f||^2.$$

Proof: We will prove the inequality above *first,* and then use it to prove that $g \in H$. By the Pythagorean theorem, for each $N \in \mathbb{N}$,

$$\| \sum_{n=1}^{N} \hat{f}_n \phi_n \|^2 = \sum_{n=1}^{N} |\hat{f}_n|^2.$$

Above, we have used the convenient notation

$$\hat{f}_n = \langle f, \phi_n \rangle.$$

We call $\hat{f}_n$ the n^{th} Fourier coefficient of f with respect to the orthonormal set (ONS) $\{\phi_n\}$. We compute that the square of the distance between f and its partial Fourier series

$$0 \le \|f - \sum_{n=1}^{N} \hat{f}_n \phi_n\|^2 = \|f\|^2 - 2\Re\langle f, \sum_{1}^{N} \hat{f}_n \phi_n \rangle + \| \sum_{1}^{N} \hat{f}_n \phi_n \|^2.$$

Let's look at the middle bit:

$$\langle f, \sum_{1}^{N} \hat{f}_n \phi_n \rangle = \sum_{1}^{N} \overline{\hat{f}_n} \langle f, \phi_n \rangle = \sum_{1}^{N} \overline{\hat{f}_n} \hat{f}_n = \sum_{1}^{n} |\hat{f}_n|^2.$$

Hence,

$$0 \leq ||f||^2 - 2\sum_{1}^{N} |\hat{f}_n|^2 + \sum_{n=1}^{N} |\hat{f}_n|^2 = ||f||^2 - \sum_{1}^{N} |\hat{f}_n|^2$$

so re-arranging

$$\sum_{1}^{N} |\hat{f}_n|^2 \leq ||f||^2.$$

Letting $N \to \infty$, we obtain the inequality

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2 \leq ||f||^2.$$

To prove that in fact

$$g \in H,$$

we will show that

$$\{F_N\}_{N\geq 1}, \quad F_N := \sum_{n=1}^{N} \hat{f}_n \phi_n$$

is a Cauchy sequence in H. Since Hilbert spaces are complete, it follows that this Cauchy sequence converges to a limit $F \in H$. So, let $\varepsilon > 0$ be given. Then, by Bessel's inequality, since

$$\sum_{1}^{\infty} |\hat{f}_n|^2 < \infty,$$

there exists $N \in \mathbb{N}$ such that

$$\sum_{N}^{\infty} |\hat{f}_n|^2 < \varepsilon^2.$$

This is because the tail of any convergent series can be made as small as we like. So, now if we have $N_1 \geq N_2 \geq N$, we estimate

$$||F_{N_1} - F_{N_2}||^2 = ||\sum_{N_2+1}^{N_1} \hat{f}_n \phi_n||^2 = \sum_{N_2+1}^{N_1} |\hat{f}_n|^2$$

$$\leq \sum_{N_2+1}^{\infty} |\hat{f}_n|^2 \leq \sum_{N}^{\infty} |\hat{f}_n|^2 < \varepsilon^2.$$

Consequently we have that for all $N_1 \geq N_2 \geq N$,

$$||F_{N_1} - F_{N_2}|| < \varepsilon.$$

This is the definition of being a Cauchy sequence. Consequently, we obtain that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \hat{f}_n \phi_n = g \in H.$$

By our infinite Pythagorean theorem, since ϕ_n are orthonormal, we also have that $\hat{f}_n \phi_n \in H$ are orhthogonal. We therefore have

$$||g||^2 = \sum_{n\geq 1} ||\hat{f}_n \phi_n||^2 = \sum_{n\geq 1} |\hat{f}_n|^2 ||\phi_n||^2 = \sum_{n\geq 1} |\hat{f}_n|^2 \leq ||f||^2.$$

1.4. **The 3 equivalent conditions to be an ONB in a Hilbert space.** Perhaps what makes the following theorem so nice is the pleasant setting of a Hilbert space, or translated directly from German, a Hilbert room. Hilbert rooms are cozy. The reason is because there is a notion of orthogonality, so it is very easy to find one's way around, much like the grid-like streets in the USA.

Theorem 9. *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be orthonormal in a Hilbert space, H. The following are equivalent:*

$$(1) \quad f \in H \text{ and } \langle f, \phi_n \rangle = 0 \forall n \in \mathbb{N} \implies f = 0.$$

$$(2) \quad f \in H \implies f = \sum_{n\in\mathbb{N}} \langle f, \phi_n \rangle \phi_n.$$

$$(3) \quad ||f||^2 = \sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2 .$$

The last of these is known as Parseval's equation. *If any of these three equivalent conditions hold, then we say that $\{\phi_n\}$ is an orthonormal basis of H.*

Proof: We shall proceed in order prove (1) $\implies$ (2), then (2) $\implies$ (3), and finally (3) $\implies$ (1). Stay calm and carry on.

First we assume statement (1) holds, and then we shall show that (2) must hold as well. Bessel's Inequality Theorem says that

$$g := \sum_{n\in\mathbb{N}} \langle f, \phi_n \rangle \phi_n \in H.$$

So, we would like to prove that in fact $g = f$, somehow using the fact that statement (1) holds true. Idea: let's try to show that $f - g = 0$. This will imply that $f = g$. To use (1) we should compute then

$$\langle f - g, \phi_n \rangle.$$

Let's do this.

$$\langle f - g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle g, \phi_n \rangle.$$

We insert the definition of g as the series,

$$\langle g, \phi_n \rangle = \langle \sum_{m\geq 1} \langle f, \phi_m \rangle \phi_m, \phi_n \rangle = \sum_{m\geq 1} \langle f, \phi_m \rangle \langle \phi_m, \phi_n \rangle = \langle f, \phi_n \rangle.$$

Above, we have used in the second equality the linearity of the inner product and the continuity of the inner product. In the third equality, we have used that $\langle \phi_m, \phi_n \rangle$ is 0 if $m \neq n$, and is 1 if $m = n$. Hence, only the term with $m = n$ survives in the sum. Thus,

$$\langle f - g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle f, \phi_n \rangle = 0, \quad \forall n \in \mathbb{N}.$$

By (1), this shows that $f - g = 0 \implies f = g$.

Next, we shall assume that (2) holds, and we shall use this to demonstrate (3). By (2),

$$f = \sum_{n\in\mathbb{N}} \hat{f}_n \phi_n, \quad \hat{f}_n := \langle f, \phi_n \rangle.$$

To obtain (3), we can simply apply our infinite dimensional Pythagorean theorem, which says that

$$||f||^2 = \sum_{n\in\mathbb{N}} ||\hat{f}_n \phi_n||^2 = \sum_{n\in\mathbb{N}} |\hat{f}_n|^2 ||\phi_n||^2 = \sum_{n\in\mathbb{N}} |\hat{f}_n|^2.$$

Finally, we assume (3) holds and use it to show that (1) must also hold. This is pleasantly straightforward. We assume that for some f in our Hilbert space, $\langle f, \phi_n \rangle = 0$ for all n. Using (3), we compute

$$||f||^2 = \sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2 = \sum_{n \in N} 0 = 0.$$

The only element in a Hilbert space with norm equal to zero is the 0 element. Thus $f = 0$.

1.5. **Exercises for the week: demonstreras.** Those exercises from [1] which shall be demonstrated are:

(1) (3.3.9) Suppose $\{\phi_n\}$ is an orthonormal basis for $\mathcal{L}^2(a, b)$. Show that for any $f, g \in \mathcal{L}^2(a, b)$

$$\langle f, g \rangle = \sum \langle f, \phi_n \rangle \overline{\langle g, \phi_n \rangle}.$$

(2) (3.3.10.c) Evaluate the following series by applying Parseval's equation to certain Fourier expansions:

$$\sum_{n \geq 1} \frac{n^2}{(n^2+1)^2}.$$

(3) (3.3.10.b) Evaluate the following series by applying Parseval's equation to certain Fourier expansions:

$$\sum_{n \geq 1} \frac{1}{(2n-1)^6}$$

(4) (3.4.3) Let D be the unit disk $\{x^2 + y^2 \leq 1\}$ and let $f_n(x, y) = (x + iy)^n$. Show that $\{f_n\}_{n \geq 0}$ is an orthogonal set in $\mathcal{L}^2(D)$, and compute $||f_n||$ for all n.

(5) (3.5.4) Find all λ so that there exists a solution $f(x)$ defined on $[0, \ell]$ to the equation

$$f'' + \lambda f = 0, \quad f'(0) = 0, \quad f(\ell) = 0.$$

(6) (EO 23) Find all solutions f on $[0, a]$ and corresponding λ to the equation:

$$f'' + \lambda f = 0, \quad f(0) = f'(0), \quad f(a) = -2f'(a).$$

(7) (4.2.1) Suppose a rod is mathematicized as the interval $[0, \ell]$, and the end at $x = 0$ is held at temperature zero while the end at $x = \ell$ is insulated. Find a series expansion for the temperature $u(x, t)$ given the initial temperature $f(x)$ and no sinks or sources.

1.6. **Exercises for the week: räkna själv.** Those exercises from [1] which one should solve are:

(1) (3.3.1) Show that if $\{f_n\}_{n \geq 1}$ are elements of a Hilbert space, H, and we have for some $f \in H$ that

$$\lim_{n \to \infty} f_n = f,$$

then for all $g \in H$ we have

$$\lim_{n \to \infty} \langle f_n, g \rangle = \langle f, g \rangle.$$

(2) (3.3.2) Show that for all f, g in a Hilbert space one has

$$||||f|| - ||g||| \leq ||f - g||.$$

(3) (3.3.10.d) Use Parseval's equation to compute

$$\sum_{n \geq 1} \frac{\sin^2(na)}{n^4}.$$

(4) (3.4.1) Show that $\{e^{2\pi i(mx+ny)}\}_{n,m\in\mathbb{Z}}$ is an orthogonal set in $\mathcal{L}^2(R)$ where R is any square whose sides have length one and are parallel to the coordinate axes.

(5) (3.4.6) Find an example of a sequence $\{f_n\}$ in $\mathcal{L}^2(0,\infty)$ such that $f_n(x) \to 0$ uniformly for all $x > 0$ but $f_n \not\to 0$ in the $\mathcal{L}^2$ norm.

(6) (3.5.7) Find all solutions f on $[0,1]$ and all corresponding λ to the equation:

$$f'' + \lambda f = 0, \quad f(0) = 0, \quad f'(1) = -f(1).$$

(7) (4.2.3) Let $f(x)$ be the initial temperature at the point x in a rod of length ℓ, mathematicized as the interval $[0,\ell]$. Assume that heat is supplied at a constant rate at the right end, in particular $u_x(\ell,t) = A$ for a constant value A, and that the left end is held at the constant temperature 0, so that $u(0,t) = 0$. Find a series expansion for the temperature $u(x,t)$ such that the initial temperature is given by $f(x)$.

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).