Theory of PDE MM8008/SF2739 Homework.

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Due date: 28th November at 23:59.[1] **Do not forget to add your name, email and Swedish personal id number** (if you have one) to your solutions.

Marks: Maximum 10 marks. To pass you need 7 or more marks.

1. Let l^2 consist of all sequences $\{x_n\}_{n=1}^{\infty}$. We will identifiy each vector $\mathbf{x} \in l^2$ with a formal series $\sum_{n=1}^{\infty} x_n e_n$ (think of $e_1 = (1, 0, 0, ...,\ e_2 = (0, 1, 0, 0, ...)$ et.c.). We equip l^2 with the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=1}^{\infty} x_n y_n$ and corresponding norm $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

a) Define the operator $T : l^2 \mapsto l^2$ according to

$$T\mathbf{x} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} a_{kn} x_k \right) e_n,$$

where

$$a_{kn} = \frac{1}{n + 2k}.$$

Show directly, without any reference to the theorem, that T is compact. That is if $\mathbf{x}^j \in l^2$ is a bounded sequence. Then there exists a subsequence such that $T\mathbf{x}^{k_j}$ converges.

[3 marks]

b) What is the dual of T? (T is demined as in a)).

[1 mark]

2. In this exercise we will motivate going from an integral on $\partial D \cap B'_\epsilon(2\epsilon)$ in (20). To that end let $\Sigma = \{(x', f(x'));\ |x'| \leq 1\}$ be the graph of the function continuously differentiable function $f(x')$ with bounded gradient.

Show that for any continuous function $g \in C(\Sigma)$ there exist a constant C_f such that

$$\int_{\Sigma} |g(x)| d\sigma(x) \leq C_f \int_{B'_1(0)} |g(x', f(x'))| dx',$$

where the constant C only depend on $\sup_{x' \in B'_1(0)} |\nabla f(x')|$ but not on g.[2]

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[1] If you email your solutions email me a PDF file that is either computer written or a scan of your handwritten solutions. Do not send me photos of your solution since they are usually very difficult to read.

[2] Here we use the standard mathematical practise to call C_f a constant even though it will be a function of $\sup_{x' \in B'_1(0)} |\nabla f(x')|$. The important thing is that for a given function f we can use the same constant for all functions $g \in C(\Sigma)$.

[3 marks]

3. Let $L(\mathcal{B},\mathcal{B})$ be the space of all bounded linear functionals $T:\mathcal{B}\mapsto\mathcal{B}$. prove that $L(\mathcal{B},\mathcal{B})$ is a Banach space under the norm

$$\|T\| = \sup_{x\in\mathcal{B},x\neq 0}\frac{\|Tx\|}{\|x\|}.$$

[3 marks]