# FOURIER ANALYSIS & METHODS 2020.03.11

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## 1. Pointwise convergence of Fourier series

**Theorem 1.1.** *Let $f$ be a $2\pi$ periodic function. Assume that $f$ is piecewise continuous on $\mathbb{R}$, and that for every $x \in \mathbb{R}$, the left and right limits of both $f$ and $f'$ exist at $x$, and these are finite. Let*

$$S_N(x) = \sum_{-N}^{N} c_n e^{inx},$$

*where*

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

*Then*

$$\lim_{N \to \infty} S_N(x) = \frac{1}{2} \left( f(x_-) + f(x_+) \right), \quad \forall x \in \mathbb{R}.$$

### 1.1. Key steps in the proof.

(1) Fix the point $x \in \mathbb{R}$.

(2) Write down the definition of

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy e^{inx}.$$

(3) Make a substitution in the integral defining the Fourier coefficients: let $t = y - x$. Then $y = t + x$. We have

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(t+x) e^{-int} dt.$$

(4) Use the periodicity to move the integral:

$$\int_{-\pi-x}^{\pi-x} f(t+x) e^{-int} dt = \int_{-\pi}^{\pi} f(t+x) e^{-int} dt.$$

Thus

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x) e^{-int} dt.$$

(5) Define the $N^{th}$ Dirichlet kernel:

$$D_N(t) = \frac{1}{2\pi} e^{-iNt} \sum_{n=0}^{2N} e^{int}.$$

(6) Remember (or if you forgot, show) two things about the Dirichlet kernel:

$$\int_{-\pi}^{0} D_N(t) dt = \frac{1}{2} = \int_{0}^{\pi} D_N(t) dt$$

and

$$D_N(t) = \frac{1}{2\pi} e^{-iNt} \frac{1 - e^{i(2N+1)t}}{1 - e^{it}} = \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1 - e^{it})}.$$

(7) Write
$$S_N(x) = \int_{-\pi}^{\pi} f(t+x) D_N(t) dt,$$
so the goal is to prove:
$$\lim_{N\to\infty} \left| S_N(x) - \frac{1}{2}\left(f(x_-) + f(x_+)\right)\right| = 0.$$
(8) Use the integration fact about the Dirichlet kernel to re-write:
$$\frac{1}{2} f(x_-) = \int_{-\pi}^{0} D_N(t) dt f(x_-), \quad \frac{1}{2} f(x_+) = \frac{1}{2} = \int_0^{\pi} D_N(t) dt f(x_+).$$
(9) Show that it now suffices to estimate:
$$\left| \int_{-\pi}^{0} D_N(t)(f(t+x) - f(x_-)) dt + \int_0^{\pi} D_N(t)(f(t+x) - f(x_+)) dt \right| \to 0$$
as $N \to \infty$.
(10) Use the second expression for the $N^{th}$ Dirichlet kernel. Based on this, define a new function
$$g(t) = \frac{f(t+x) - f(x_-)}{1 - e^{it}}, \qquad \text{for } t < 0,$$
$$g(t) = \frac{f(t+x) - f(x_+)}{1 - e^{it}}, \qquad \text{for } t > 0.$$
(11) Show that $g$ is piecewise continuous and piecewise differentiable. Show that $g$ is bounded.
(12) Show that one is in fact estimating $c_N(g)$, the $N^{th}$ Fourier coefficient of $g$ minus $c_{-N-1}(g)$, the $-N-1$ Fourier coefficient of $g$.
(13) Use Bessel's inequality to prove that these coefficients both tend to zero as $N \to \infty$.

## 2. Fourier coefficients of a function and its derivative

**Theorem 2.1.** *This time in Swedish for fun! Låt $f$ vara en $2\pi$-periodisk funktion med $f \in C^1(\mathbb{R})$. Sedan Fourierkoefficienterna $c_n$ av $f$ och Fourierkoefficienterna $c'_n$ av $f'$ uppfyller*
$$c'_n = inc_n.$$

### 2.1. Key steps.

(1) Use the definition of the Fourier coefficient of $f'$, $c'_n$. Write it down.
(2) Integrate by parts: move the derivative from $f'$ to the $e^{-inx}$.
(3) Use the fact that $f$, $f'$, and $e^{inx}$ are $2\pi$ periodic to kill off the boundary terms. The result should be $c'_n = inc_n$.

## 3. The 3 equivalent conditions to be an ONB in a Hilbert space

**Theorem 3.1.** *Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara ortonormala i ett Hilbert-rum, $H$. Följande tre är ekvivalenta:*
$$(1) \quad f \in H \text{ och } \langle f, \phi_n\rangle = 0 \forall n \in \mathbb{N} \implies f = 0.$$
$$(2) \quad f \in H \implies f = \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n.$$
$$(3) \quad ||f||^2 = \sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2.$$

3.1. **Key steps.**

(1) Assume that (1) is true and use it to prove (2). To do this, use Bessel's Inequality Theorem to say that
$$g := \sum_{n\geq 1} \langle f, \phi_n\rangle \phi_n \in H.$$
(2) Next, compute
$$\langle g - f, \phi_n\rangle, \quad \text{and show it is zero for all } n.$$
(3) Assume now that (2) is true and use it to prove (3). To do this, use the infinite dimensional Pythagorean theorem and the fact that $\{\phi_n\}$ are orthonormal.
(4) Assume now that (3) is true and use it to prove (1).

## 4. The Best Approximation Theorem

**Theorem 4.1.** *Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara en ortonormal mängd i ett Hilbert-rum, $H$. Om $f \in H$,*
$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n|| \leq ||f - \sum_{n\in\mathbb{N}} c_n\phi_n||, \quad \forall \{c_n\}_{n\in\mathbb{N}} \in \ell^2,$$
*och $=$ gäller $\iff$ $c_n = \langle f, \phi_n\rangle$ gäller $\forall n \in \mathbb{N}$.*

4.1. **Key steps.**

(1) Define
$$g := \sum \widehat{f_n}\phi_n, \quad \widehat{f_n} = \langle f, \phi_n\rangle,$$
and
$$\varphi := \sum c_n \phi_n.$$
(2) A clever trick:
$$||f - \varphi||^2 = ||f - g + g - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 + 2\Re\langle f - g, g - \varphi\rangle.$$
(3) Prove that
$$\langle f - g, g - \varphi\rangle = 0.$$
To do this, just pop in the definitions of $g$ and $\varphi$ and use the properties about scalar products (which you MUST MEMORIZE!!).
(4) After this calculation we get
$$||f - \varphi||^2 = ||f - g + g - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 \geq ||f - g||^2,$$
with equality if and only if
$$||g - \varphi||^2 = 0.$$
(5) Use the Pythagorean Theorem to conclude that
$$||g - \varphi||^2 = 0 \iff \widehat{f_n} = c_n \quad \forall n \in \mathbb{N}.$$

## 5. Cute properties of SLPs

**Theorem 5.1** (Cute facts about SLPs)**.** *Let $f$ and $g$ be eigenfunctions for a regular SLP in an interval $[a, b]$ with weight function $w(x) > 0$. Let $\lambda$ be the eigenvalue for $f$ and $\mu$ the eigenvalue for $g$. Then:*

*(1) $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$;*
*(2) If $\lambda \neq \mu$, then:*
$$\int_a^b f(x)\overline{g(x)}w(x)dx = 0.$$

(1) $L$ is self-adjoint means that
$$\langle Lf, f\rangle = \langle f, Lf\rangle.$$
(2) Use the fact that $Lf = -\lambda w f$ and the properties of scalar products (which you have memorized!!!) in the above equality to show that $\lambda \in \mathbb{R}$.
(3) For the second part use the self adjoint-ness and the eigenvalue equation to investigate
$$\langle Lf, g\rangle.$$

## 6. The big bad convolution approximation theorem

**Theorem 6.1.** *Let $g \in L^1(\mathbb{R})$ such that*
$$\int_{\mathbb{R}} |g(x)| dx = 1.$$
*Define*
$$\alpha = \int_{-\infty}^0 g(x) dx, \quad \beta = \int_0^\infty g(x) dx.$$
*Assume that $f$ is piecewise continuous on $\mathbb{R}$ and its left and right sided limits exist for all points of $\mathbb{R}$. Assume that either $f$ is bounded on $\mathbb{R}$ or that $g$ vanishes outside of a bounded interval. Let, for $\epsilon > 0$,*
$$g_\epsilon(x) = \frac{g(x/\epsilon)}{\epsilon}.$$
*Then*
$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}.$$

### 6.1. **Key steps.**

(1) Fix the point $x$.
(2) Show that it is enough to prove that
$$\lim_{\epsilon \to 0} \int_{-\infty}^0 f(x-y) g_\epsilon(y) dy - \int_{-\infty}^0 f(x+) g(y) dy = 0$$
and also
$$\lim_{\epsilon \to 0} \int_0^\infty f(x-y) g_\epsilon(y) dy - \int_0^\infty f(x-) g(y) dy = 0.$$
The argument is same for both, so choose one. I choose the first one.
(3) Do a substitution in the second integral, setting $z = \epsilon y$, so $y = z/\epsilon$, and $dz/\epsilon = dy$. This shows that:
$$\int_{-\infty}^0 \left( f(x-y) g_\epsilon(y) - f(x+) g(y) \right) dy = \int_{-\infty}^0 g_\epsilon(y) \left( f(x-y) - f(x+) \right) dy.$$
(4) Now, to estimate
$$\int_{-\infty}^0 g_\epsilon(y) \left( f(x-y) - f(x+) \right) dy,$$
split the integral into $\int_{-\infty}^{-\delta} + \int_{-\delta}^0$.
(5) First estimate
$$\int_{-\delta}^0 g_\epsilon(y) \left( f(x-y) - f(x+) \right) dy,$$
using the definition of $f(x+)$ as the right hand limit. This fixes the value of $\delta$.
(6) Next estimate
$$\int_{-\infty}^{-\delta} g_\epsilon(y) \left( f(x-y) - f(x+) \right) dy.$$
Do this for each of the two cases separately.

## 7. The Fourier inversion formula

This theory item is really a julklapp. All one must know is the Fourier inversion formula.

**Theorem 7.1** (FIT)**.** *Assume that $f \in L^2(\mathbb{R})$. Define the Fourier transform to be:*
$$\hat{f}(\xi) = \int_{\mathbb{R}} f(y) e^{-iy\xi} dy.$$
*Then as an equality in $L^2(\mathbb{R})$ we have*
$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y) e^{ixy} dy.$$

♡

## 8. Plancharel's Theorem

This one is also on the light side.

**Theorem 8.1.** *Assume $f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$. With the Fourier transform defined by*

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx,$$

*then we have*

$$\langle \hat{f}, \hat{g} \rangle = \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi = 2\pi \langle f, g \rangle = 2\pi \int_{\mathbb{R}} f(x)\overline{g(x)}dx,$$

*and*

$$\int_{\mathbb{R}} |\hat{f}(x)|^2 dx = ||\hat{f}||^2 = 2\pi ||f||^2 = 2\pi \int_{\mathbb{R}} |f(x)|^2 dx.$$

### 8.1. Key steps.

(1) Start on the right side.
(2) Use the FIT to write $f$ in terms of its Fourier transform.
(3) Use the magic of complex conjugation to obtain the Fourier transform of $g$.

## 9. The Sampling Theorem

**Theorem 9.1.** *Let $f \in L^2(\mathbb{R})$. Assume that there is $L > 0$ so that $\hat{f}(\xi) = 0$ $\forall \xi \in \mathbb{R}$ with $|\xi| > L$, then:*

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{L}\right) \frac{\sin(n\pi - tL)}{n\pi - tL}.$$

### 9.1. Key steps.

(1) Expand $\hat{f}(x)$ in a Fourier series on the interval $[-L, L]$

$$\hat{f}(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L}\hat{f}(x)dx.$$

(2) Use the FIT to write

$$f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixt}\hat{f}(x)dx = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt}\hat{f}(x)dx.$$

(3) Substitute the Fourier expansion of $\hat{f}$ into this integral,

$$f(t) = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt}\sum_{-\infty}^{\infty} c_n e^{in\pi x/L}dx.$$

(4) Compute the Fourier coefficients

$$c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L}\hat{f}(x)dx = \frac{1}{2L}\int_{\mathbb{R}} e^{ix(-n\pi/L)}\hat{f}(x)dx = \frac{2\pi}{2L} f\left(\frac{-n\pi}{L}\right).$$

(5) Substitute back into $f(t)$,

$$f(t) = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt}\sum_{-\infty}^{\infty}\frac{\pi}{L} f\left(\frac{-n\pi}{L}\right) e^{in\pi x/L}dx.$$

(6) Swap the sum and the integral

$$f(t) = \frac{1}{2L}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\int_{-L}^{L} e^{x(it-in\pi/L)}dx.$$

(7) Compute:

$$\int_{-L}^{L} e^{x(it-in\pi/L)}dx = \frac{e^{L(it-in\pi/L)}}{i(t-n\pi/L)} - \frac{e^{-L(it-in\pi/L)}}{i(t-n\pi/L)} = \frac{2i}{i(t-n\pi/L)}\sin(Lt-n\pi).$$

(8) Substitute back inside.

## 10. The generating function for the Bessel functions

**Theorem 10.1.** *For all $x$ and for all $z \neq 0$, the Bessel functions, $J_n$ satisfy*

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}(z - \frac{1}{z})}.$$

### 10.1. Key steps.

(1) Write out the Taylor series expansion for the exponential functions:

$$e^{xz/2} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!},$$

and

$$e^{-x/(2z)} = \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!}.$$

(2) Multiply these together:

$$e^{xz/2} e^{-x/(2z)} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!} \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!} = \sum_{j,k \geq 0} (-1)^k \left(\frac{x}{2}\right)^{j+k} \frac{z^{j-k}}{j!k!}.$$

(3) We need a sum over $\mathbb{Z}$ but we just have two sums over $j, k \geq 0$. To get this, define the variable

$$n = j - k.$$

Write everything in terms of $n$ and $k$, which gives

$$e^{xz/2} e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{z^n}{\Gamma(n+k+1)k!}.$$

(4) Recognize that the sum over $k$ is the definition of $J_n(x)$.

## 11. Orthogonality of the Hermite polynomials

**Theorem 11.1.** *The Hermite polynomials $\{H_n\}_{n=0}^{\infty}$ are orthogonal on $\mathbb{R}$ with respect to the weight function $w(x) = e^{-x^2}$. Recall here that*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

*and so the statement is that*

$$\int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = 0, \quad n \neq m.$$

### 11.1. Key steps.

(1) Assume WLOG that

$$n > m \geq 0.$$

(2) Do integration by parts on $\langle H_n, H_m \rangle$:

$$(-1)^n \int_{\mathbb{R}} \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_m(x) dx = (-1)^n \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right) H_m(x) \Big|_{x=-\infty}^{\infty}$$
$$+ (-1)^{n+1} \int_{\mathbb{R}} \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right) H_m'(x) dx.$$

(3) Use the fact that

$$\frac{d^{n-1}}{dx^{n-1}} e^{-x^2} = \text{ polynomial times } e^{-x^2}$$

and the fact that $e^{-x^2}$ goes to zero faster as $|x| \to \infty$ than any polynomial (Godzilla!) to conclude that

$$(-1)^n \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right) H_m(x) \Big|_{x=-\infty}^{\infty} = 0.$$

(4) Show inductively that you can do this $n$ times to get

$$(-1)^n \int_{\mathbb{R}} \left( \frac{d^n}{dx^n} e^{-x^2} \right) H_m(x) dx = (-1)^{n+n} \int_{\mathbb{R}} e^{-x^2} \left( \frac{d^n}{dx^n} H_m(x) \right) dx.$$

(5) If one differentiates $H_m$, a polynomial of degree $m < n$, $n$ *times,* the result is zero. So the integral on the right is just zero.

## 12. The generating function for the Hermite polynomials

**Theorem 12.1.** *For any $x \in \mathbb{R}$ and $z \in \mathbb{C}$, the Hermite polynomials,*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

*satisfy*

$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{2xz - z^2}.$$

### 12.1. Key steps.

(1) Start with

$$e^{-(x-z)^2} = e^{-x^2 + 2xz - z^2}.$$

(2) Compute the power series expansion with respect to $z$ around $z = 0$,

$$e^{-(x-z)^2} = \sum_{n \geq 0} a_n z^n,$$

where

$$a_n = \frac{1}{n!} \frac{d^n}{dz^n} e^{-(x-z)^2}, \quad \text{evaluated at } z = 0.$$

(3) Compute the coefficients using the chain rule with the variable $u = x - z$, thus

$$\begin{aligned} a_n &= \frac{1}{n!} \frac{d^n}{du^n} e^{-u^2} \left( \frac{du}{dz} \right)^n \bigg|_{z=0 \implies u=x} \\ &= \frac{1}{n!} \frac{d^n}{dx^n} e^{-x^2} (-1)^n. \end{aligned}$$

(4) Pop it back into the Taylor series expansion:

$$e^{-(x-z)^2} = e^{-x^2 + 2xz - z^2} = \sum_{n \geq 0} \frac{z^n}{n!} (-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

(5) Multiply both sides by $e^{x^2}$.