The Dirichlet and Neumann Problem in General Domains.

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1 Introduction.

In this part of the course we will investigate the possibility to solve the Dirichlet and Neumann problems in rather general domains. There are several reasons for this. First of all it is a very important problem - in particular for applied problems. But my main reason is that it is a difficult problem that will force us to develop an abstract approach to mathematics. So far, most of the theory in the course have been explicit. We construct solutions by means of greens functions et.c. But if we have a complicated domain there is no hope to be able to write down a formula for the solution. Instead we will develop some functional analytic machinery that will help us to show that solutions exist.

The story is interesting and the method powerful. Also, I hope that it will be very educational to see an abstract theory developed from scratch.

In this notes **we will always assume that the space dimension** $n \geq 3$. This assumption is not really necessary, but since the Newtonian kernel is logarithmic at infinity when $n = 2$ the theory looks somewhat different in $\mathbb{R}^2$. My main goal in the last part of the course is to make a transition into really abstract, modern if you like, mathematics and not to provide an encyclopedia of the existence results for the Laplace equation. Therefore is it reasonable to consider either the case $n = 2$ or the case $n \geq 3$ and the case $n \geq 3$ seems to be the more reasonable choice.

Also since I am mostly interested in the transition to abstract mathematics I have mostly provided proofs for the Dirichlet problem when the corresponding proofs for the Neumann problem are similar. I believe that understanding the Dirichlet case is enough to understand how the theory fits together and that is more important than having all the proofs in detail.

2 Overview

You have seen how to solve the Dirichlet problem

$$\begin{array}{ll} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D, \end{array} \tag{1}$$

in the special case when $D = B_1(0)$ or $D = \mathbb{R}^n_+$. In this part of the course we will investigate the existence of solutions to the Dirichlet problem (1) in more general domains D. We will also show existence of solutions to the Neumann problem:

$$\begin{array}{ll} \Delta u = 0 & \text{in } D \\ \frac{\partial u}{\partial \nu} = f & \text{on } \partial D, \end{array} \tag{2}$$

where ν is the outer normal of the domain D.

In order to find solutions to the Dirichlet problem (1) and the Neumann problem (2) it is reasonable to try to adjust the methods that we used to solve the Dirichlet problem in the domain $\mathbb{R}^n_+$. The solution to the Dirichlet problem in $\mathbb{R}^n_+$ with boundary data[1] $f(y')$ was given by the integral formula

$$u(x) = \frac{1}{\omega_n}\int_{\mathbb{R}^{n-1}} \frac{2x_n}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}} f(y')dy'. \tag{3}$$

We will indeed take the approach of trying to write down an integral formula like the one in (3), but there will be an interesting twist that will give us the opportunity to look at how functional analysis methods are used in PDE theory.

Before we formally state the main area of investigation in this part of the course we will have to define the domains D that will be of interest. Throughout these notes all domains D will be bounded!

Definition 2.1. *We say, for $0 < \alpha \leq 1$, that a function $f \in C^{1,\alpha}$, at times $C^{1,\alpha}(\Sigma)$ if we want to specify the domain of definition Σ, if f is continuously differentiable in its domain of definition Σ and if*

$$\|f\|_{C^{1,\alpha}} := \sup_{x\in\Sigma}|f(x)| + \sup_{x\in\Sigma}|\nabla(x)| + \sup_{x,y\in\Sigma}\frac{|\nabla f(x)-\nabla f(y)|}{|x-y|^\alpha} < \infty \tag{4}$$

We say that an open set $D \subset \mathbb{R}^n$ is a $C^{1,\alpha}-$domain if, for every $x_0 \in \partial D$, there exists a coordinate system $(y_1, y_2, ..., y_n)$ and an $r > 0$ such that $\partial D \cap B_r(x_0)$ is the graph $f(y') = y_n$ of some function $f \in C^{1,\alpha}$.

We say that something depends on the $C^{1,\alpha}$ character of the domain if it depends only on the dimension n, the $r > 0$ and the maximal $C^{1,\alpha}-$norm, as defined in (4), of f in the definition of $C^{1,\alpha}$ domain.

The main theorem we will prove over the next couple of lectures is

Theorem 2.1. *Given a bounded $C^{1,\alpha}-$domain D and a continuous function $f \in C(\partial D)$ then there exists a unique solution u to the Dirichlet problem*

$$\begin{aligned} \Delta u &= 0 && \text{in } D \\ u &= f && \text{on } \partial D. \end{aligned} \tag{5}$$

We will also prove a similar theorem for the Neumann problem (2). As it turns out, in order to prove the existence of solutions to the Dirichlet problem in D we will have to prove existence in the domain D^c as well and existence of the Neumann problem in D and D^c. So we solve four problems simultaneously!

2.1 Strategy and Outline of the Lectures.

We may write the solution to the Dirichlet problem in $\mathbb{R}^n_+$ given by equation (3) in the following form

$$u(x) = \int_{\mathbb{R}^{n-1}} \frac{\partial N(x,y)}{\partial \nu_y} f(y')dy', \tag{6}$$

[1] We will often use the notation $x' = (x_1, x_2, ..., x_{n-1})$, $y' = (y_1, ..., y_{n-1})$ et.c. for the first $n-1$ coordinates.

where $\nu_y = -e_n$ is the outward pointing normal and

$$N(x,y) = \frac{-1}{(n-2)\omega_n}\frac{1}{|x-y|^{n-2}} \tag{7}$$

is the Newtonian kernel[2] (fundamental solution) for the Laplace equation.

Since we are interested in $C^{1,\alpha}-$domains that have a well defined normal ν at every point of the domain we may guess (which is a good, but not exactly right, guess) that the *double layer potential* defined according to

$$u(x) = \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} f(y) d\sigma(y) \tag{8}$$

is a solution to the Dirichlet problem (1) in an arbitrary $C^{1,\alpha}-$domain D. In (8) the notation $\frac{\partial N(x,y)}{\partial \nu_y}$, obviously, stands for the normal derivative of the Newtonian kernel with respect to the normal of D and $\sigma(y)$ is the area measure of the boundary ∂D.

Considering the Neumann problem (2) one might guess that the solution should be, *the single layer potential defined according to*

$$u(x) = \int_{\partial D} N(x,y) f(y) d\sigma(y). \tag{9}$$

At least if the guess (8) is correct for the Dirichlet problem; since then the Neumann data of the function in (9) should be, after differentiating under the integral sign,

$$\frac{\partial u(x)}{\partial \nu_x} = \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_x} f(y) d\sigma(y). \tag{10}$$

The good thing with these guesses is that we can actually calculate the Dirichlet and Neumann data of the functions defined in (8) and (9). It is always a very powerful tool in mathematics to be able to calculate things explicitly - even when, as in this case, it turns out that the calculation leads to an answer that we did not expect.

2.2 Calculation of the potentials.

We begin by showing that the functions defined by (8) and (9) are indeed harmonic.

Proposition 2.1. *Let D be a bounded domain then the functions defined by (8) and (9) are harmonic in $\mathbb{R}^n \setminus \partial D$.*

Proof: Fix an $x_0 \in \mathbb{R}^n \setminus \partial D$ then $N(x,y)$ is $C^\infty(B_{2\epsilon}(x_0))$ in the x variable for every $y \in \partial D$; here we choose $\epsilon > 0$ according to $3\epsilon = \text{dist}(x_0, \partial D)$. This means that all $x-$derivatives of $N(x,y)$ are uniformly continuous and equicontinuous

[2] Throughout these Notes we will assume that $n \geq 3$ in order not to have to consider the case $n = 2$ when the Newtonian kernel is logarithmic. The theory in $\mathbb{R}^2$ is very similar.

for $x \in B_\epsilon(x_0)$ and $y \in \partial D$ (here we use that ∂D is bounded and therefore compact). We may therefore differentiate under the integral sign in (8) and (9) which leads to, in the case of (8),

$$\Delta u(x_0) = \int_{\partial D} \frac{\partial \Delta_x N(x_0, y)}{\partial \nu_y} f(y) d\sigma(y) = 0,$$

since $N(x, y)$ is harmonic for $x \neq y$. A similar calculation shows that u defined by (9) is harmonic. □

Next we calculate the boundary values on ∂D of the functions (8) and (9). As it turns out the solutions does not have the boundary values that we would hope. The calculations are also rather involved so we begin with some lemmata.

Lemma 2.1. *Let Σ be a piece of C^1 surface, with C^1 boundary, not intersecting the origin. We also assume that each ray through the origin only intersect Σ in one point. Then*

$$\int_\Sigma \frac{\partial N(0, y)}{\partial \nu_y} d\sigma(y) = \frac{\alpha}{\omega_n}, \tag{11}$$

where α is the solid angle of the cone of rays from the origin through the surface Σ.[3]

Proof: Let D be the cone consisting of all straight lines from the origin through the points of the surface Σ and set $D_\epsilon = D \setminus B_\epsilon(0)$ where $\epsilon > 0$ is so small that $B_\epsilon(0) \cap \Sigma = \emptyset$.

Then, since $N(0, y)$ is harmonic away from the origin,

$$0 = \int_{D_\epsilon} \Delta_y N(0, y) dy = \int_\Sigma \frac{\partial N(0, y)}{\partial \nu_y} d\sigma(y) + \int_{\partial B_\epsilon(0) \cap D} \frac{\partial N(0, y)}{\partial \nu_y} d\sigma(y), \tag{12}$$

where we used an integration by parts in the last inequality. Notice that the surface integrals in (12) over $\partial D_\epsilon \setminus (\Sigma \cup \partial B_\epsilon(0))$ vanishes since $\frac{\partial N(0,y)}{\partial \nu_y} = 0$ on that part of the boundary.

From (12) it follows that

$$\int_\Sigma \frac{\partial N(0, y)}{\partial \nu_y} d\sigma(y) = -\int_{\partial B_\epsilon(0) \cap D} \frac{\partial N(0, y)}{\partial \nu_y} d\sigma(y) =$$

$$= \frac{1}{\omega_n} \int_{\partial B_\epsilon(0) \cap D} \frac{1}{\epsilon^{n-1}} d\sigma(y) = \frac{\alpha}{\omega_n},$$

where we have used that $\frac{\partial}{\partial \nu_y} = -\frac{y}{|y|} \cdot \nabla$ in the second equality. □

Next we need a lemma that controls

$$\frac{\partial N(x, y)}{\partial \nu_y} = -\frac{1}{(n-2)\omega_n} \nu_y \cdot \nabla_y \frac{1}{|x - y|^{n-2}} = \frac{1}{\omega_n} \frac{\nu_y \cdot (x - y)}{|x - y|^n} \tag{13}$$

[3]The solid angle of a cone is just the area of the unit sphere intersected with the cone.

for $x, y \in \partial D$. By shifting the coordinate system we may assume that $x = 0$ and $\nu_0 = e_n$. Furthermore, if D is a $C^{1,\alpha}-$domain (which we assume) then ∂D consists of the graph of a $C^{1,\alpha}$ function $f(y')$ in a small neighborhood of the origin. In these coordinates and with this f we may write (the absolute value of) the expression in (13) as

$$\left|\frac{1}{\omega_n}\frac{(-\nabla' f, 1)\cdot(y', f(y'))}{\sqrt{1+|\nabla' f|^2}}\frac{1}{|y|^n}\right| \le C\frac{|y'||\nabla' f(y')|+|f(y')|}{|y|^n} \le$$

$$\le C\frac{1}{|y'|^{n-1-\alpha}}, \tag{14}$$

for y small, say $|y| \le c_0$. The constant to the right in (14) will depend on the $C^{1,\alpha}$ norm of f and also on the dimension - that is on the domain D but not on the points $x, y \in \partial D$. Clearly (14) also holds for any $|y| > c_0$ if the domain is bounded, although we may have to increase the constant C for it to hold for arbitrary $y \in \partial D$. But even if we increase C, it will still only depend on the domain D. We have therefore shown the following lemma.

Lemma 2.2. *Given a bounded $C^{1,\alpha}-$domain D there is a constant C_D that depend on the dimension and on the domain D, but not on $x, y \in \partial D$, such that*

$$\left|\frac{\partial N(x,y)}{\partial \nu_y}\right| = \left|\frac{1}{(n-2)\omega_n}\nu_y\cdot\nabla_y\frac{1}{|x-y|^{n-2}}\right| \le \frac{C_D}{|x-y|^{n-1-\alpha}.} \tag{15}$$

From the estimate (15) we may derive the following estimate.

Lemma 2.3. *Given a bounded $C^{1,\alpha}-$domain D there exist constants $C_1, c_0 > 0$ depending on D such that for every $x_0 \in \partial D$ and every $r < c_0$ and $x \in B_r(x_0)\cap D$*

$$\int_{B_r(x_0)\cap\partial D}\left|\frac{(x-y)\cdot\nu_y}{|x-y|^n}\right| d\sigma(y) \le C_1.$$

Proof: We let $\epsilon > 0$ be so small that $\partial D \cap B_{2\epsilon}(x_0)$ is the graph of some function. Fix $x \in B_\epsilon(x_0)\cap D$ then there exists some $\bar{x} \in \partial D$ such that x lies on the normal line of ∂D through $\bar{x}$. To see this we just consider the largest ball $B_\delta(x) \subset D$, then $\overline{B_\delta(x)}\cap\partial D \ne \emptyset$ so there is an $\bar{x} \in \partial B_\delta(x)\cap\partial D$ and the normal of of ∂D at $\bar{x}$ will coincide with a radii of $B_\delta(x)$. We fix $\bar{x}$ and $\delta = |x-\bar{x}|$ as above.

Let us compute, for $y \in \partial D$,

$$|x-y|^2 = |\bar{x}-y|^2 - 2(\bar{x}-y)\cdot(\bar{x}-x) + |x-\bar{x}|^2. \tag{16}$$

Since $\bar{x}-x$ points in the direction of the normal of ∂D and ∂D is $C^{1,\alpha}$ it follows (as in (14)) that

$$|(\bar{x}-y)\cdot(\bar{x}-x)| = \delta|(\bar{x}-y)\cdot\nu_{\bar{x}}| \le C\delta|\bar{x}-y|^{1+\alpha}, \tag{17}$$

where we also used that $|\bar{x}-x| = \delta$.

From (16) and (17) we may conclude that

$$|x-y|^2 \geq |\bar{x}-y|^2 + \delta^2 - C\delta|\bar{x}-y|^{1+\alpha} \geq \frac{1}{4}(|\bar{x}-y|^2+\delta^2),$$

provided that $|\bar{x}-y| \leq \epsilon$ is small enough.

Next we use the triangle inequality, and that ∂D is $C^{1,\alpha}$, to estimate

$$|(x-y)\cdot\nu_y| \leq |(\bar{x}-y)\cdot\nu_y| + |(x-\bar{x})\cdot\nu_y| \leq |(\bar{x}-y)\cdot\nu_y| + \delta \leq C|\bar{x}-y|^{1+\alpha} + \delta. \quad (18)$$

Using (17) and (18) we may deduce that

$$\frac{|(x-y)\cdot\nu_y|}{|x-y|^n} \leq C\frac{|\bar{x}-y|^{1+\alpha}+\delta}{(|\bar{x}-y|+\delta)^n}$$

and therefore

$$\int_{B_\epsilon(x_0)\cap\partial D} \frac{|(x-y)\cdot\nu_y|}{|x-y|^n} d\sigma(y) \leq \quad (19)$$

$$\leq \int_{B_\epsilon(x_0)\cap\partial D} \frac{C}{|\bar{x}-y|^{n-1-\alpha}} d\sigma(y) + C\delta \int_{B_\epsilon(x_0)\cap\partial D} \frac{1}{(|\bar{x}-y|+\delta)^n} d\sigma(y).$$

The first integral to the right in (19) is convergent since the exponent in the denominator is less than $n-1$. Furthermore, we may estimate the second integral in (19) by noticing that $B_\epsilon(x_0) \subset B_{2\epsilon}(\bar{x})$ and therefore

$$C\delta \int_{B_\epsilon(x_0\cap\partial D)} \frac{1}{(|\bar{x}-y|+\delta)^n} d\sigma(y) \leq$$

$$\leq C\delta \int_{B_{2\epsilon}(\bar{x})\cap\partial D} \frac{1}{(|\bar{x}-y|+\delta)^n} d\sigma(y) \leq C_1\delta \int_0^{2\epsilon} \frac{dr}{(r+\delta)^2} dr < C_2, \quad (20)$$

where we changed to polar coordinates in the second inequality and used that $\delta < \epsilon$ (since $B_\delta(x) \subset D$ and $x \in B_\epsilon(x_0)$ for an $x_0 \in \partial D$). Using the last estimate in (19) leads to

$$C\delta \int_{B_\epsilon(x_0\cap\partial D)} \frac{1}{(|\bar{x}-y|+\delta)^n} d\sigma(y) \leq C,$$

where C is independent of x. This proves the lemma. □

Theorem 2.2. The Dirichlet problem: *Let $f \in C(\partial D)$ and u be defined by the integral (8) in the domain D. Then u may be extended to a continuous function $u \in C(\overline{D})$ and*

$$u(x) = \frac{1}{2}f(x) + \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} f(y) d\sigma(y) \quad (21)$$

on ∂D.

The Neumann Problem: *Let $f \in C(\partial D)$ and u be defined by the integral (9) in the domain D. Then u may be extended to a continuous function $u \in C(\overline{D})$ with well defined normal derivative:*

$$\frac{\partial u(x)}{\partial \nu_x} = -\frac{1}{2} f(x) + \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_x} f(y) d\sigma(y) \tag{22}$$

on ∂D.

Proof: We will only prove the theorem for the Dirichlet case, the Neumann case is similar.

We split the integral, for $x \in \partial D$ and $z \in D$, into

$$\int_{\partial D} \frac{\partial N(z,y)}{\partial \nu_y} f(y) d\sigma(y) = f(x) \int_{\partial D \cap B_\epsilon(x)} \frac{(z-y)\cdot \nu_y}{\omega_n |z-y|^n} d\sigma(y) +$$

$$+ \int_{\partial D \cap B_\epsilon(x)} (f(y) - f(x)) \frac{(z-y)\cdot \nu_y}{\omega_n |z-y|^n} d\sigma(y) + \tag{23}$$

$$+ \int_{\partial D \setminus B_\epsilon(x)} f(y) \frac{(z-y)\cdot \nu_y}{\omega_n |z-y|^n} d\sigma(y) = I_1 + I_2 + I_3.$$

We begin by estimating I_1. By Lemma 2.1 it follows that

$$I_1 = f(x) \frac{\alpha(z,x,\epsilon)}{\omega_n}, \tag{24}$$

where $\alpha(z,x,\epsilon)$ is the solid angle of the cone with vertex in z and $\partial D \cap B_\epsilon(x)$ as basis. Clearly,

$$I_1 = f(x) \frac{\alpha(z,x,\epsilon)}{\omega_n} \to \frac{1}{2} f(x) \tag{25}$$

as $z \to x$ and $\epsilon \to 0$.

In order to estimate I_2 we use Lemma 2.3

$$|I_2| \le \sup_{y \in B_\epsilon(x) \cap \partial D} (|f(y) - f(x)|) \left| \int_{B_\epsilon(x) \cap \partial D} \frac{(z-y)\cdot \nu_y}{|z-y|^n} d\sigma(y) \right| \le \tag{26}$$

$$\le C \sup_{y \in B_\epsilon(x) \cap \partial D} |f(y) - f(x)|.$$

Clearly, by sending $z \to x$ and $\epsilon \to 0$ it follows from (26) and continuity of f that $I_2 \to 0$.

Also, sending $\epsilon \to 0$ it follows, from Lemma 2.2, that

$$I_3 \to \int_{\partial D} f(y) \frac{(z-y)\cdot \nu_y}{\omega_n |z-y|^n} d\sigma(y).$$

The estimates of I_1, I_2 and I_3 implies the Theorem. □

A similar calculation shows that we may extend the potential to a continuous function in $\overline{D^c}$. The proof is very similar to the proof of Theorem 2.2 and therefore omitted.

Corollary 2.1. The Dirichlet problem: *Let $f \in C(\partial D)$ and u be defined by the integral (8) in the domain D^c. Then u may be extended to a continuous function $u \in C(\overline{D^c})$ and*

$$u(x) = \frac{1}{2} f(x) - \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} f(y) d\sigma(y) \tag{27}$$

on ∂D, here ν_y is the exterior normal of D.

The Neumann Problem: *Let $f \in C(\partial D^c)$ and u be defined by the integral (9) in the domain D^c. Then u may be extended to a continuous function $u \in C(\overline{D^c})$ with well defined normal derivative:*

$$\frac{\partial u(x)}{\partial \nu_x} = \frac{1}{2} f(x) + \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_x} f(y) d\sigma(y) \tag{28}$$

on ∂D, here ν_x is the exterior normal of D.

We end this section by indicating that we may recover f if we know the single layer potential.

Corollary 2.2. *Let u be defined by (9) and let ν_+ be the outer normal of Ω^c and ν_- be the outer normal of Ω then*

$$f(x) = \frac{\partial u}{\partial \nu_+} - \frac{\partial u}{\partial \nu_-} \qquad \text{on } \partial\Omega.$$

2.3 The Strategy to show Existence of Solutions.

It follows directly from Theorem 2.2 and Proposition 2.1 that if we can find a solution ϕ to the integral equation

$$f(x) = \frac{1}{2}\phi(x) + \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} \phi(y') d\sigma(y)$$

then the function

$$u(x) = \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_y} \phi(y) d\sigma(y)$$

will be harmonic by Proposition 2.1 and, by Theorem 2.2, satisfy the Dirichlet boundary values $u(x) = f(x)$ on ∂D. A similar reasoning reduces the Neumann problem to finding a solution ϕ to the integral equation

$$f(x) = -\frac{1}{2}\phi(x) + \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu_x} \phi(y') d\sigma(y).$$

Therefore we will define the kernels

$$K(x,y) = \frac{\partial N(x,y)}{\partial \nu_y} \tag{29}$$

and

$$K^*(x,y) = \frac{\partial N(x,y)}{\partial \nu_x} \tag{30}$$

and the operators

$$\left(\frac{1}{2}I + T\right)\phi(x) = \frac{1}{2}\phi(x) + \int_{\partial D} K(x,y)\phi(y)d\sigma(y) \tag{31}$$

and

$$\left(-\frac{1}{2}I + T^*\right)\phi(x) = -\frac{1}{2}\phi(x) + \int_{\partial D} K^*(x,y)\phi(y)d\sigma(y), \tag{32}$$

where I is the identity operator.

Knowing that solving the Dirichlet problem involves finding a ϕ such that $\frac{1}{2}\phi + T\phi = f$ and solving the Neumann problem involves finding a ϕ such that $-\frac{1}{2}\phi + T^*\phi = f$ our main goal will be to investigate the operators T and T^*. In particular, if we want to show that the Dirichlet (or Neumann) problem is solvable for every $f \in C(\partial D)$ it is enough to show that $\frac{1}{2}I + T$ (and $-\frac{1}{2}I + T^*$ respectively) is surjective onto $C(\partial D)$. We will do this in several steps.

Step 1: In order to define the operators $\frac{1}{2}I + T$ and $-\frac{1}{2}I + T^*$ we need to agree on a domain of definition. The operators $\frac{1}{2}I + T$ and $-\frac{1}{2}I + T^*$ will certainly not be surjective onto the set of continuous functions if we restrict them to a to small domain of definition. In order to properly investigate the domains of definition we need to introduce Banach and Hilbert spaces. The Banach and Hilbert spaces are vector spaces of functions that has good abstract properties that will allow us to analyze not only the spaces but also operators defined on them.

Step 2: Next we will investigate a certain class of operators called compact operators. Compact operators plays an important role in functional analysis. The aim is to develop Riesz-Shauder theory for operators (the Riesz-Schauder Theory is sometimes called Fredholm theory and we will use both names interchangeably). This is a powerful tool to investigate the relationship between the range and null space of operators and their adjoints. But we are getting ahead of ourselves now and will have to wait before we define those concepts.

Step 3: With the Riesz-Shauder Theory at hand we will show that the operators T and T^* falls under that theory and using that theory we will show that there is a solution to the Dirichlet problem for arbitrary boundary data $f \in C(\partial D)$ and that, under a natural condition, the Neumann problem also have a solution.

2.4 Exercises

1. In this exercise we will motivate going from an integral on $\partial D \cap B'_\epsilon(2\epsilon)$ in (20). To that end let $\Sigma = \{(x', f(x'));\ |x'| \leq 1\}$ be the graph of the function continuously differentiable function $f(x')$ with bounded gradient.

Show that for any continuous function $g \in C(\Sigma)$ there exist a constant C_f such that

$$\int_\Sigma |g(x)|d\sigma(x) \leq C_f \int_{B_1'(0)} |g(x', f(x'))|dx',$$

where the constant C only depend on $\sup_{x' \in B_1'(0)} |\nabla f(x')|$ but not on g.[4]

2. In Definition 2.1 we said that the $C^{1,\alpha}-$character of D depends on the least $r > 0$ in the definition of $C^{1,\alpha}-$domain. Prove that for any bounded $C^{1,\alpha}-$domain we may always cover ∂D by a finite number of balls $B_r(x_j)$ for $x_j \in \partial D$ with r independent of x_j.

3 Banach and Hilbert Spaces.

In this section we will define Banach and Hilbert spaces and derive some of the properties of those spaces that we need. It is obvious that the domain of definition of an operator will determine its range. But it is a more subtle relationship between the structural properties of the domain of definition and the properties of the operator. One of the basic structures of a space is the Banach space.

3.1 Definition of a Banach Space.

We need to define several simple concepts in order to define a Banach space. First of all we need to define a linear space.

Definition 3.1. *We say that a set A is a linear space over $\mathbb{R}$ if*

1. *A is a commutative group. That is there is an operation "$+$" defined on $A \times A \mapsto A$ such that*
   (a) *For any $u, v, w \in A$ the following holds: $u + v = v + u$ (addition is commutative), $(u + v) + w = u + (v + w)$ (addition is associative).*
   (b) *There exists an element $0 \in A$ such that for all $u \in A$ we have $u + 0 = u$.*
   (c) *For every $u \in A$ there exists an element $v \in A$ such that $u + v = 0$, we usually denote $v = -u$.*
2. *There is an operation (multiplication) defined on $\mathbb{R} \times A \mapsto A$ such that*
   (a) *For all $a, b \in \mathbb{R}$ and $u, v \in A$ we have $a \cdot (u + v) = a \cdot u + a \cdot v$ and $(a + b) \cdot u = a \cdot u + b \cdot u$.*
   (b) *For all $a, b \in \mathbb{R}$ and $u \in A$ we have $(ab) \cdot u = a \cdot (b \cdot u)$.*

[4]Here we use the standard mathematical practise to call C_f a constant even though it will be a function of $\sup_{x' \in B_1'(0)} |\nabla f(x')|$. The important thing is that for a given function f we can use the same constant for all functions $g \in C(\Sigma)$.

Examples: 1: The most obvious example is if $A = \mathbb{R}^n$ and "$+''$ is normal vector addition and "$\cdot''$ is normal multiplication by a real number.

2: Another example that will be much more important to us is if A is a set of functions, say the set of functions with two continuous derivatives on Ω. Clearly all the above assumptions are satisfied for twice continuously differentiable functions if we interpret "$+''$ and "$\cdot''$ as the normal operations.

Many linear spaces satisfies another important structure: that we can measure distances. Distances allow us to talk about convergence and to do analysis. We will only be interested in spaces where we have a norm.[5]

Definition 3.2. *A norm $\|\cdot\|$ on a linear space A is a function from $A \mapsto \mathbb{R}$ such that the following axioms are satisfied:*

1. *For any $u \in A$ we have $\|u\| \geq 0$ with equality if and only if $u = 0$ (The Positivity Axiom).*
2. *For any $u, v \in A$ we have $\|u+v\| \leq \|u\| + \|v\|$ (The Triangle Inequality).*
3. *For any $u \in A$ and $a \in A$ we have $\|a \cdot u\| = |a|\|u\|$ (The Homogeneity Axiom).*

If a linear space A has a norm we say that A is a normed linear space, or just a normed space.

At times we will use $\|u\|_A$ to indicate that we are using the $A-$norm.[6]

Examples: 1: The linear space $\mathbb{R}^n$ is a normed space with norm $\|(u_1, ..., u_n)\| = (u_1^2 + u_2^2 + ... + u_n^2)^{1/2}$.

2: The set of continuous functions on $[0, 1]$ is a normed space under the norm

$$\|u\| = \int_0^1 |u(x)| dx.$$

3: If we define

$$\|u\|_{C^2(\Omega)} = \sup_{x\in\Omega} |u(x)| + \sup_{x\in\Omega} |\nabla u(x)| + \sup_{x\in\Omega} |D^2 u(x)|, \tag{33}$$

Then the set of two times continuously differentiable functions $u(x)$ on Ω for which $\|u\|_{C^2(\Omega)}$ is finite forms a normed space: $C^2(\Omega)$. Notice that $\frac{1}{x} \notin C^2(0,1)$ even though $\frac{1}{x}$ is continuous with continuous derivatives on $(0, 1)$.

The final property that we need in our function-spaces is completeness.

Definition 3.3. *Let A be a normed linear space. Then we say that A is complete if every Cauchy sequence $u^j \in A$ converges to an element $u \in A$.*

[5] There is a slight difference between a norm and a distance function (metric). Every norm $\|\cdot\|$ defines a distance according to $d(x, y) = \|x - y\|$. But not every distance function defines a norm. The norm can be viewed as a distance that respects the linear space structure.

[6] This is important when we are working with several spaces simultaneously, in particular if we are considering elements that lay in different normed spaces.

Remember that we say that $u^j \in A$ is a Cauchy sequence if there for every $\epsilon > 0$ exists a N_ϵ such that $\|u^j - u^k\| < \epsilon$ for all $j, k > N_\epsilon$. So if A is complete and u^j is a Cauchy sequence in A then there should exist an element $u^0 \in A$ such that $\lim_{j\to\infty} \|u^j - u^0\| = 0$.

Examples: 1: It is an easy consequence of the the Bolzano-Weierstrass theorem that $\mathbb{R}^n$ is complete. In particular, every Cauchy sequence is bounded. Therefore the Bolzano-Weierstrass theorem implies that it has a convergent subsequence. That the Cauchy condition implies that the entire sequence converges to the same limit is easy to see.

2: The space of continuous functions on $[0,1]$ with norm $\|u\| = \int_0^1 |u(x)|dx$ is not complete. For instance if

$$u^j(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2} - \frac{1}{j} \\ \frac{j}{2}\left(x - \left(\frac{1}{2} - \frac{1}{j}\right)\right) & \text{if } \frac{1}{2} - \frac{1}{j} < x < \frac{1}{2} + \frac{1}{j} \\ 1 & \text{if } \frac{1}{2} + \frac{1}{j} \le x \le 1 \end{cases}$$

then u^j is continuous and forms a Cauchy sequence. However, the pointwise limit is clearly

$$u^0(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1}{2} & \text{if } x = \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

But u^0 is not continuous and therefore not in the space of continuous functions on $[0,1]$. Therefore that space is not complete.

However, if we consider the space $C([0,1])$ of continuous functions with norm

$$\|u\|_{C([0,1])} = \sup_{x\in[0,1]} |u(x)|$$

then we get a complete space. This since the limit $\lim_{j\to\infty} u^j(x)$ is uniform and continuity is preserved under uniform limits.

It is important to notice that the properties of the space is dependent on the norm. Continuous functions with an integral norm are not complete, but continuous spaces with a supremum norm are complete.

3: The space $C^2(\Omega)$ with norm defined by the supremum as in (33) is also a complete space.

Clearly, in order to do analysis on a linear space it is desirable that the linear space is complete. We therefore make the following definition.

Definition 3.4. *We call a complete normed linear space is a Banach space.*

3.2 The Definition of a Hilbert Space.

Another structural property that makes $\mathbb{R}^n$ easy to work with is that we may measure angles between vectors $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ by means of the formula

$$\arccos\left(\frac{a \cdot b}{|a||b|}\right).$$

The important thing here is that $\mathbb{R}^n$ has a scalar product.

For some Banach spaces it is possible to define a scalar product that allows us to define orthogonality between elements of the Banach space. But before we investigate which spaces that have an inner product we need to axiomatize the properties that we expect an inner product to have.

Definition 3.5. *Let $\mathcal{H}$ be a Banach space. We say that a function $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \mapsto \mathbb{R}$ is an inner product if*

1. $(x, x) \geq 0$ *for all* $x \in \mathcal{H}$ *and* $(x, x) = 0$ *if and only if* $x = 0$.
2. $(x, y) = (y, x)$ *for all* $x, y \in \mathcal{H}$
3. $$(\alpha x + \beta x', y) = \alpha(x, y) + \beta(x', y)$$
   for all $x, x', y \in \mathcal{H}$ *and* $\alpha, \beta \in \mathbb{R}$.

Example and Definition of l^2: Let l^2 be the space of all sequences of real numbers $a = (a_1, a_2, a_3, ...)$, $a_k \in \mathbb{R}$ such that

$$\|a\| = \left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2} < \infty.$$

If $a = (a_1, a_2, ...)$ and $b = (b_1, b_2, ...)$ then we may define an inner product on l^2 according to

$$(a, b) = \sum_{k=1}^{\infty} a_k b_k.$$

Example and Definition of L^2: If we define the space $L^2([0, 1])$ to consist of all (Lebesgue) integrable functions on $[0, 1]$ such that $\int_0^1 |f(x)|^2 dx < \infty$. Then the following defines an inner product on $L^2([0, 1])$

$$(f, g) = \int_0^1 f(x)g(x)dx.$$

More generally we may define $L^2(D)$ to be the space of all functions $f(x)$[7] defined on D such that

$$\|f\|_{L^2(D)} = \left(\int_D |f(x)|^2 dx\right)^{1/2} < \infty. \tag{34}$$

[7]Lebesgue measurable functions to be exact.

Definition 3.6. *A Hilbert space $\mathcal{H}$ is a Banach space with an inner product defined such that*
$$\|x\| = |(x,x)|^{1/2}.$$

The examples above of l^2 and $L^2([0,1])$ are Hilbert spaces.[8]

3.3 Schauder bases for Banach and Hilbert spaces.

To show that every Banach space has a basis we need to resort to Zorn's Lemma. Zorn's Lemma is the consequence of the Axiom of Choice and transfinite induction. The proof is usually given in courses on the foundations of mathematics and we will not give it here. Before we state Zorn's Lemma we need a definition.

Definition 3.7. *We say that a set P is partially ordered if there is a binary relation $\leq$ defined on a subset of pairs of P so that $\leq$ satisfies*[9]

1. *if $a \leq b$ and $b \leq c$ then $a \leq c$*
2. *if $a \leq b$ and $b \leq a$ then $a = b$*
3. *$a \leq a$ for all $a \in P$.*

Lemma 3.1. [Zorn's Lemma] *Let P be a partially ordered set under the order relation $\leq$ and assume that for every chain*
$$p_1 \leq p_2 \leq p_3 \leq ...$$
there exists an element $p \in P$ such that $p_j \leq p$ for all p_j in the chain. Then there exists a maximal element $p_{max} \in p$ such that $p_{max} \leq p_0$ if and only if $p_0 = p_{max}$.

We can now prove that every Banach space has a Basis.

Proposition 3.1. *Every Banach space $\mathcal{B}$ has a basis. That is there exists a set of basis vectors $b_\alpha \in \mathcal{B}$ such that every element $b \in \mathcal{B}$ can be written, for some $N \in \mathcal{N}$, in a unique way as*
$$b = \sum_{j=1}^{N} a_j b_{\alpha_j},$$
where b_{α_j} are basis vectors and $a_j \in \mathbb{R}$.

[8] One of the most important reasons to introduce the Lebesgue integral is that $L^2(D)$ becomes complete if we interpret the integral in (34) in the Lebesgue sense. We will not use any specific properties of the Lebesgue integral in this course except that $L^2(D)$ is complete. It is therefore not important that you know exactly what the Lebesgue integral is in the rest of the course. Just think of the Riemann integral and assume that $L^2(D)$ is complete.

[9] Notice that for a partial order there might exist two elements $x, y \in P$ such that neither $x \leq y$ nor $y \leq x$ holds. For instance if P consists of all sequences of real numbers than we may define the partial order $a \leq b$ if $a_k \leq b_k$ for all k. With this order, if $a = (1, 0, 0...)$ and $b = (0, 1, 0, ...)$ then neither $a \leq b$ nor $b \leq a$.

Proof: Let P be the set of all sub-sets of linearly independent vectors of $\mathcal{B}$. Then P is a partially ordered set under the ordering $p \leq q$ if $p \subset q$ for any sets $p, q \in P$. By Zorn's Lemma it follows that there is a set $p_{max} \in P$. Since $p_{max} \in P$ it is linearly independent. But it must also span $\mathcal{B}$ since if it does not, say $b_0 \notin \mathrm{Span}(p_{max})$, then $p_{max} \cup \{b_0\} \in P$ and $p_{max} \leq p_{max} \cup \{b_0\}$ which contradicts the maximality of p_{max}. This finishes the proof. □

3.4 Linear operators on Banach Spaces.

Since we intend to view the Dirichlet and Neumann problems as mappings from the boundary data to the solutions of the problems it is important to develop some theory for linear mappings between Banach and Hilbert spaces. We begin with a definition.

Definition 3.8. *We say that an operator $T : \mathcal{B}_1 \mapsto \mathcal{B}_2$ between two Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ is linear if*

$$T(\alpha x + \beta y) = \alpha T x + \beta T y$$

for all $x, y \in \mathcal{B}_1$ and $\alpha, \beta \in \mathbb{R}$.

We say that a linear operator $T : \mathcal{B}_1 \mapsto \mathcal{B}_2$ is bounded if there exist a constant C_0 such that

$$\|Tx\|_{\mathcal{B}_2} \leq C_0 \|x\|_{\mathcal{B}_1} \tag{35}$$

for all $x \in \mathcal{B}_1$.

We also define the operator norm of a bounded linear operator $T : \mathcal{B}_1 \mapsto \mathcal{B}_2$ by

$$\|T\| = \sup_{x \in \mathcal{B}_1} \frac{\|Tx\|_{\mathcal{B}_2}}{\|x\|_{\mathcal{B}_1}},$$

that is $\|T\|$ is the least constant C_0 such that (35) holds for all $x \in \mathcal{B}_1$.

Linear operators have nice properties, for instance boundedness is equivalent to continuity.

Proposition 3.2. *A linear operator $T : \mathcal{B}_1 \mapsto \mathcal{B}_2$ between two Banach spaces is continuous if and only if it is bounded.*

Proof: If $T : \mathcal{B}_1 \mapsto \mathcal{B}_2$ is bounded and $x_j \in \mathcal{B}_1$ converges to x_0 then x_j forms a Cauchy sequence. Since

$$\|Tx_j - Tx_k\|_{\mathcal{B}_2} \leq C_0 \|x_j - x_k\|$$

and x_j is Cauchy it follows that Tx_j is a Cauchy sequence in $\mathcal{B}_2$ and therefore $Tx_j \to y_0$ for some $y_0 \in \mathcal{B}_2$, here we use that $\mathcal{B}_2$ is complete (as is every Banach space) by assumption. We need to show that $Tx_0 = y_0$. This follows from

$$\|Tx_0 - y_0\|_{\mathcal{B}_2} \leq \|Tx_j - Tx_0\|_{\mathcal{B}_2} + \|Tx_j - y_0\|_{\mathcal{B}_2} \leq C\|x_j - x_0\|_{\mathcal{B}_1} + \|Tx_j - y_0\|_{\mathcal{B}_2} \to 0,$$

since $x_j \to x_0$ in $\mathcal{B}_1$ and $Tx_j \to y_0$ in $\mathcal{B}_2$. It follows that every bounded operator is continuous.

To prove that continuous operators are bounded we show that if T is not bounded then T is not continuous. If T is not bounded then there exists a sequence x_j such that $\|x_j\|_{B_1} \leq \frac{1}{j}$ but $\|Tx_j\|_{\mathcal{B}_2} = 1$. Since $T0 = 0$ by linearity and $\|Tx_j\|_{\mathcal{B}_2} = 1$ it follows that $Tx_j \not\to T0 = 0$ even though $x_j \to 0$. It follows that T is not continuous. $\square$

We continue by showing a first invertability result for linear mappings.

Proposition 3.3. *let $T : \mathcal{B} \mapsto \mathcal{B}$ be a bounded linear mapping from the Banach-space $\mathcal{B}$ to itself. Assume furthermore that $\|T\| = \lambda < 1$. Then $I - T$, where I is the identity operator, is invertable and*

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k, \tag{36}$$

where we interpret $T^0 = I$.

Proof: Fix $K > 1$. Then

$$(I-T)\sum_{k=0}^{K} T^k x = (I-T)(I+T+T^2+...+T^K)x = x - T^{K+1}x. \tag{37}$$

But since $\|T\| = \lambda < 1$

$$\|T^{K+1}x\| \leq \lambda\|T^K x\| \leq \lambda^2\|T^{K-1}x\| \leq ... \leq \lambda^{K+1}\|x\|. \tag{38}$$

Since $\lambda < 1$ it follows that $\sum_{k=0}^{K} T^k x$ is a Cauchy sequence in K and, since $\mathcal{B}$ is complete, it converges.

From (37) and (38) it follows that

$$\lim_{K\to\infty}(I-T)\sum_{k=0}^{K} T^k x - x = \lim_{K\to\infty} T^k x = 0,$$

therefore $\sum_{k=0}^{\infty} T^k$ is indeed the inverse of $I - T$. $\square$

3.5 Riesz Representation Theorem and Duals in Hilbert Spaces.

In this section we will gather some information of dual operators and prove an important theorem called the Riesz representation Theorem.

Definition 3.9. *Let $T : \mathcal{H} \mapsto \mathcal{H}$ be a linear operator on a Hilbert space $\mathcal{H}$. Then we say that an operator T^* is the adjoint of T if*

$$(Tx, y) = (x, T^*y)$$

for all $x, y \in \mathcal{H}$.

Theorem 3.1. [Riesz Representation Theorem] *Let $\mathcal{H}$ be a Hilbert space and $F : \mathcal{H} \mapsto \mathbb{R}$ be a bounded linear operator. Then there exists a unique element $f \in \mathcal{H}$ such that*

$$F(x) = (x, f).$$

Proof: If $F(x) = 0$ for all $x \in \mathcal{H}$ then $f = 0$ satisfies the properties of the theorem. So let us assume that $F \neq 0$. Then there exist a vector $z \notin \mathrm{Ker}(F)$, we may even choose z orthogonal to $\mathrm{Ker}(F)$.

Then, since F is linear,

$$F\left(x - \frac{F(x)}{F(z)}z\right) = F(x) - \frac{F(x)}{F(z)}F(z) = 0,$$

that is $x - \frac{F(x)}{F(z)}z \in \mathrm{Ker}(F)$.

Since z is orthogonal to the kernel of K it follows that

$$\left(x - \frac{F(x)}{F(z)}z, z\right) = 0 \Rightarrow (x, z) = \frac{F(x)}{F(z)}\|z\|^2.$$

That is

$$F(x) = \left(x, \frac{F(z)}{\|z\|^2}z\right).$$

This implies the existence of an $f = \frac{F(z)}{\|z\|^2}z$ as in the conclusion of the theorem.

To show that the element f is unique we assume that there exists two $f_1, f_2 \in \mathcal{H}$ such that $F(x) = (x, f_1) = (x, f_2)$. Then $(x, f_1 - f_2) = 0$ for all $x \in \mathcal{H}$ which in particular implies, with $x = f_1 - f_2$, that

$$0 = (f_1 - f_2, f_1 - f_2) = \|f_1 - f_2\|^2,$$

that is $f_1 = f_2$. □

Proposition 3.4. *The operators T and T^* defined in (31) and (32) are adjoint operators on $L^2(\partial D)$ (which justifies our notation).*

Proof: Pick $f, g \in L^2(\partial D)$ then

$$(Tf, g) = \int_{\partial D}(Tf(x))g(x)d\sigma(x) = \int_{\partial D}\left[\int_{\partial D}\frac{\partial N(x,y)}{\partial \nu_y}f(y)d\sigma(y)\right]g(x)d\sigma(x).$$

Applying Fubini's Theorem (change of order of integration) to this implies that

$$(Tf, g) = \int_{\partial D}\left[\int_{\partial D}\frac{\partial N(x,y)}{\partial \nu_y}g(x)d\sigma(x)\right]f(y)d\sigma(y). \tag{39}$$

We need to show that the integral in the square brackets in (39) is equal to

$$T^*g(y) = \int_{\partial D}\frac{\partial N(y,x)}{\partial \nu_y}g(x)d\sigma(x).$$

This follows since
$$\frac{\partial N(x,y)}{\partial \nu_y} = -\frac{\nu_y}{\omega_n}\cdot\frac{x-y}{|x-y|^n}$$
which is the same as
$$\frac{\partial N(y,x)}{\partial \nu_y} = -\frac{\nu_y}{\omega_n}\cdot\frac{x-y}{|x-y|^n}.$$
We may therefore continue (39)
$$(Tf,g) = \int_{\partial D}\left[\int_{\partial D}\frac{\partial N(x,y)}{\partial \nu_y}g(x)d\sigma(x)\right]f(y)d\sigma(y) =$$
$$= \int_{\partial D}(T^*g(y))f(y)d\sigma(y) = (f,T^*g).$$
Since $f,g \in L^2(\partial D)$ where arbitrary it follows that T^* is the adjoint of T. □

3.6 Exercises

1. State and prove the Riesz Representation Theorem in $\mathbb{R}^n$.
2. Find the dual of the following linear operators
   (a) $A : \mathbb{R}^n \mapsto \mathbb{R}^n$ when A is an $n \times n$ matrix.
   (b) $L : L^2((0,1)) \mapsto L^2(0,1)$ when L is given by $Lf(x) = \int_0^x f(t)dt$.
3. Later we will use that $L^2(\partial D)$ has a countable Schauder basis. Try to convince yourself that any function $f \in L^2(\partial D)$ may be written $f(x) = \sum_{k=1}^{\infty} a_k \chi_{\Omega_k}(x)$ where $a_k \in \mathbb{R}$ and
$$\chi_{\Omega_k}(x) = \begin{cases} 1 & \text{if } x \in \Omega_k \\ 0 & \text{if } x \notin \Omega_k, \end{cases}$$
   and the sets Ω_k are the balls $B_{r_k}(x_k)$ where x_k runs over all $x \in \mathbb{Q}^n$ and $r_k \in \mathbb{Q}$ and $r_k > 0$. In particular, the dimension of the vector space $L^2(\partial D)$ is at most countable.
4. Prove that $C(D)$ is a Banach space with the norm $\|f\| = \sup_{x\in D}|f(x)|$. Prove that $C^{1,\alpha}(D)$ is a Banach space.
5. Prove the triangle inequality in l^2:
$$\|a+b\|_{l^2} \le \|a\|_{l^2} + \|b\|_{l^2}.$$
6. Prove that if $T_K : L^2(D) \mapsto L^2(D)$ is given by
$$T_K f(x) = \int_D K(x,y)f(y)dy$$
   then $T^*_{K(x,y)} = T_{K(y,x)}$.

7. Let $L(\mathcal{B},\mathcal{B})$ be the space of all bounded linear functionals $T : \mathcal{B} \mapsto \mathcal{B}$. prove that $L(\mathcal{B},\mathcal{B})$ is a Banach space under the norm

$$\|T\| = \sup_{x\in\mathcal{B}} \frac{\|Tx\|}{\|x\|}.$$

8. Let $\|\cdot\|_{\mathcal{B}}$ be a norm on a Banach space $\mathcal{B}$. Show that the function

$$d(x,y) = \frac{\|x-y\|}{1+\|x-y\|}$$

defines a metric on $\mathcal{B}$.[10] Is $d(x,0)$ a norm on $\mathcal{B}$?

9. Let (x,y) be an inner product on $\mathcal{H}$. Prove that $\|x\| = \sqrt{(x,x)}$ satisfies the assumptions for being a norm on $\mathcal{H}$.

4 Linear Operators and Riesz-Schauder Theory.

Given a Banach (or Hilbert) space we are interested in mappings $T : \mathcal{B} \mapsto \mathcal{B}$. In particular to show when (if) an operator T is surjective. Let us begin with some examples to get a feel for the kind of operators that appears in analysis.

Example 1: Let $\mathcal{B} = C([0,1])$ and define $T : \mathcal{B} \mapsto \mathcal{B}$ according to

$$T\phi(x) = \int_0^x \phi(t)dt.$$

Then clearly T is defined on $\mathcal{B}$ (since every continuous function is integrable). But the range of T consists of the subspace of $C^1([0,1])$ consisting of functions that vanish at the origin. It follows that T is not surjective.

Example 2: Let $\mathcal{B} = C^2(B_1(0))$ and define $T : \mathcal{B} \mapsto \mathcal{B}$ to be the operator that $Tf = u$ where u solves

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } B_1(0) \\ u(x) = 0 & \text{on } \partial B_1(0). \end{array}$$

The operator is well defined since the Dirichlet problem is solvable in the unit ball.

Example 3: Let $\mathcal{H} = L^2(\mathbb{R})$ and let $\mathcal{F} : \mathcal{H} \mapsto \mathcal{H}$ be the Fourier transform

$$\mathcal{F}f(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx.$$

By the Fourier inversion theorem $\mathcal{F}$ is invertable and therefore surjective.

[10] A metric is a function $d : \mathcal{B}\times\mathcal{B} \mapsto \mathbb{R}$ such that i) $d(x,y) \geq 1$ with equality iff $x = y$, ii) $d(x,y) = d(y,x)$ and iii) $d(x,z) \leq d(x,y) + d(x,z)$.

Example 4: Let $\mathcal{H} = l^2$ where l^2 is the space of all sequences $a = (a_1, a_2, a_3, ...)$, $a_k \in \mathbb{R}$, with the norm

$$\|a\|_{l^2} = \left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2}.$$

The space l^2 is a Hilbert space with inner product $(a, b) = \sum_{k=1}^{\infty} a_k b_k$. We may, for every bounded sequence $t_1, t_2, ... \in \mathbb{R}$ define the operator $T : l^2 \mapsto l^2$ by $Ta = (t_1 a_1, t_2 a_2, t_3 a_3, ...)$. Then T is a bounded operator $T : l^2 \mapsto l^2$ and if there exists constants $0 < c \le C$ such that $c \le |t_k| \le C$ then T is invertable and therefore surjective.

Let us consider the operators $\frac{1}{2}I + T$ and $-\frac{1}{2}I + T^*$ defined in (31) and (32). We will consider the operators to be defined on $L^2(\partial D)$; that is $T : L^2(\partial D) \mapsto L^2(\partial D)$ and $T^* : L^2(\partial D) \mapsto L^2(\partial D)$. Both operators are of the form[11] $\frac{1}{2}I + S$ where I is the identity mapping. Therefore we will consider T and T^* to be a small perturbations of (half) the identity operator. The question is in what sense the operator S is small: we will show that it is small in the sense that it is compact.

Definition 4.1. *We say that an operator between a Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, $T : \mathcal{B}_1 \mapsto \mathcal{B}_2$, is compact if for every bounded sequence $f_j \in \mathcal{B}_1$ the sequence $Tf_j \in \mathcal{B}_2$ has a convergent subsequence.*

We need to develop some theory for compact operator in order to show that T and T^* are compact. We begin by showing that compact operators are closed under convergence in norm.

Proposition 4.1. *Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \mapsto \mathcal{H}$. Assume furthermore that there exists a sequence of compact linear operators $T_k : \mathcal{H} \mapsto \mathcal{H}$ such that $\|T - T_k\| \to 0$ then T is also compact.*

Proof: We have to show that given a bounded sequence $x_j \in \mathcal{H}$ then for every $\epsilon > 0$ there exists a subsequence x_{l_j} such that if $j, m > J_\epsilon$ then $\|Tx_{l_j} - Tx_{l_m}\| < \epsilon$. For that we assume that we have a sequence x_j, such that $\|x_j\| \le M$, and pick an $\epsilon > 0$. Since $T_k \to T$ there exists a K_ϵ such that

$$\|T - T_k\| \le \frac{\epsilon}{3M} \quad \text{for all } k > K_\epsilon. \tag{40}$$

Next we fix a $k_0 > K_\epsilon$. Since T_{k_0} is compact $T_{k_0} x_j$ has a convergent, and therefore Cauchy, sub-sequence $T_{k_0} x_{l_j}$. It follows that there exists an J_ϵ such that if $j, m > J_\epsilon$ then

$$\|T_{k_0} x_{l_j} - T_{k_0} x_{l_m}\| < \frac{\epsilon}{3} \quad \text{for all } j, m > J_\epsilon. \tag{41}$$

[11] Possibly after an inconsequential multiplication by -1.

It follows that for $j, m > J_\epsilon$

$$\|Tx_{l,j} - Tx_{l_m}\| \leq \|Tx_{l_j} - T_{k_0}x_{l_j}\| + \|Tx_{l_m} - T_{k_0}x_{l_m}\| + \|T_{k_0}x_{l_j} - T_{k_0}x_{l_m}\| <$$

$$< \|T - T_{k_0}\|\|x_{l_j}\| + \|T - T_{k_0}\|\|x_{l_m}\| + \frac{\epsilon}{3} < \frac{\epsilon\|x_{l_j}\|}{3M} + \frac{\epsilon\|x_{l_m}\|}{3M} + \frac{\epsilon}{3} \leq \epsilon,$$

where we used (41) and the definition of operator norm in the first strict inequality, (40) together with $k_0 > K$ in the second strict inequality and that $\|x_j\| \leq M$ in the final inequality. □

Example 5: Any bounded operator $T : \mathcal{B} \mapsto \mathcal{B}$ such that Range(T) is finite dimensional is compact. This follows since if $x_j \in \mathcal{B}$ is bounded then Tx_j is bounded (since T is a bounded operator). This means that Tx_j is a bounded sequence in a finite dimensional vector space and we may use the Bolzano-Weierstrass Theorem in finite dimensional spaces to find a convergent sub-sequence.

Definition 4.2. *We define the rank of an operator $T : \mathcal{B}_1 \mapsto \mathcal{B}_2$ between two Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ to be the dimension of the range of T:*

$$\mathit{Rank}(T) = \mathit{Dim}(\mathit{Range}(T)).$$

Corollary 4.1. *If $T_k \to T$ and each T_k is has finite rank then T is compact.*

Proof: This follows from Proposition 4.1 since all operators T_k has finite rank and are therefore compact. □

Proposition 4.2. *An operator $T : \mathcal{H} \mapsto \mathcal{H}$ from a Hilbert space to itself is compact if and only if T^* is compact.*

Proof: Since $(T^*)^* = T$; that is

$$\big((T^*)^*x, y\big) = \big(x, T^*y\big) = \big(Tx, y\big)$$

for all $x, y \in \mathcal{H}$, it is enough to show that if T is compact so is T^*.

We begin by showing that if $x_n \in \mathcal{H}$ is bounded and T is compact then T^*x_n is bounded; that is if T is compact then T^* is bounded. Let $x_n \in \mathcal{H}$ be a sequence such that $\|x_n\| \leq M$ for some $M \in \mathbb{R}$ then

$$\|T^*x_n\|^2 = (T^*x_n, T^*x_n) = (x_n, T(T^*x_n)) \leq \tag{42}$$

$$\leq \|x_n\|\|T(T^*x_n)\| \leq M\|T\|\|T^*x_n\|.$$

Dividing both sides of (42) by $\|T^*x_n\|$ implies that $\|T^*\| \leq \|T\|$ so T^* is bounded.

Next we need to show that T^*x_n has a convergent subsequence. It is enough, by the completeness of Banach spaces, to show that T^*x_n contains a Cauchy sequence. First we notice that since T is compact by assumption and T^*x_n is bounded the sequence $T(T^*x_n)$ has a convergent sub-sequence $T(T^*x_{n_k})$. Now we may calculate, as in (42),

$$\|T^*x_{n_k} - T^*x_{n_l}\|^2 = \big(x_{n_k} - x_{n_l}, T(T^*x_{n_k}) - T(T^*x_{n_l})\big) \leq \tag{43}$$

$$\leq 2M\|T(T^*(x_{n_k}) - T^*(x_{n_l})\| \to 0,$$

since $T(T^*x_{n_k})$ is convergent. But (43) implies that $T^*x_{n_k}$ is a Cuahcy sequence. □

Definition 4.3. *We say that a function $K : \Sigma \times \Sigma \mapsto \mathbb{R}$ is a Hilbert-Schmidt kernel if*

$$\|K\|_{L^2(\Sigma\times\Sigma)} = \left(\int_\Sigma \int_\Sigma |K(x,y)|^2 d\sigma(x) d\sigma(y)\right)^{1/2} < \infty.$$

Given a Hilbert-Schmidt kernel K we say that the operator $T_K : L^2(\Sigma) \mapsto L^2(\Sigma)$ defined by

$$T_K f = \int_\Sigma K(x,y) f(y) d\sigma(y)$$

is a Hilbert-Schmidt operator.

Theorem 4.1. *If T_K is a Hilbert-Schmidt operator then T_K is compact and*

$$\|T_K\| \leq \|K\|_{L^2(\Sigma\times\Sigma)}.$$

Proof: We use the Hölder inequality to estimate[12]

$$|T_K f(x)| \leq \int_\Sigma |K(x,y)||f(y)| d\sigma(y) \leq \left(\int_\Sigma |K(x,y)|^2 d\sigma(y)\right)^{1/2} \|f\|_{L^2(\Sigma)}. \tag{44}$$

If we square (44) and integrate with respect to x we arrive at

$$\|T_K f\|^2_{L^2(\Sigma)} \leq \|K\|^2_{L^2(\Sigma\times\Sigma)} \|f\|^2_{L^2(\Sigma)}. \tag{45}$$

It follows that T_K is bounded form $L^2(\Sigma)$ to $L^2(\Sigma)$. By Corollary 4.1 the proof is complete if we can approximate T_K by a sequence of operators T_j with finite dimensional range.

In order to do this we choose an orthonormal basis[13] $\psi_k(x)$ of $L^2(\Sigma)$. It follows that we may write

$$K(x,y) = \sum_{k,l=1}^\infty a_{kl} \psi_k(x) \psi_l(y).$$

[12] Technically this estimate is only valid for almost every x since $\int |K(x,y)|^2 dy$ might diverge for some x - but only for a set of measure zero.

[13] Here we actually use that we may choose a countable basis. We will not prove that the basis is countable in this course. However, see Exercise 3 from the previous section.

By orthonormality of the basis ψ_k

$$\|K(x,y)\|^2_{L^2(\Sigma\times\Sigma)} =$$

$$= \int_\Sigma \int_\Sigma \left(\sum_{k,l=1}^{\infty} a_{kl}\psi_k(x)\psi_l(y) \right) \left(\sum_{\alpha,\beta=1}^{\infty} a_{\alpha\beta}\psi_\alpha(x)\psi_\beta(y) \right) d\sigma(x)d\sigma(y) =$$

$$= \sum_{k,l=1}^{\infty} |a_{kl}|^2.$$

We may therefore approximate T_K by the integral operator T_j where T_j is defined by

$$T_j f(x) = \int_\Sigma \sum_{k,l=1}^{j} a_{kl}\psi_k(x)\psi_l(y) f(y) d\sigma(y).$$

Clearly the range of T_j, which is spanned by $\psi_1, \psi_2, ..., \psi_j$ is finite dimensional. Furthermore, by the estimate (45) with $T_K - T_j$ in place of T_K,

$$\|T_K - T_j\| = \left\| K - \sum_{k,l=1}^{j} a_{kl}\psi_k(x)\psi_l(y) \right\|_{L^2(\Sigma\times\Sigma)} = \left(\sum_{k,l>j}^{\infty} |a_{kl}|^2 \right)^{1/2} \to 0$$

as $j \to \infty$ since the series $\sum_{k,l=1}^{\infty} |a_{kl}|^2$ is convergent. Therefore $T_j \to T_K$ and each T_j has finite dimensional range. It follows, from Corollary 4.1, that T_K is compact. $\square$

In order to show that T and T^* are compact operators we will need the following lemma.

Lemma 4.1. *Let $K(x,y)$ be a kernel that satisfies*

$$\sup_{x\in\Sigma} \int_\Sigma |K(x,y)| d\sigma(y) \le C \text{ and } \sup_{y\in\Sigma} \int_\Sigma |K(x,y)| d\sigma(x) \le C.$$

Then if T_K is the operator defined, on $L^2(\Sigma)$, according to

$$T_K f(x) = \int_\Sigma K(x,y) f(y) d\sigma(y)$$

then

$$\|T_K f\|_{L^2(\Sigma)} \le C\|f\|_{L^2(\Sigma)}.$$

Proof: Notice that by Hölder's inequality

$$|Tf(x)| \le \left(\int_\Sigma |K(x,y)| d\sigma(y) \right)^{1/2} \left(\int_\Sigma |K(x,y)||f(y)|^2 d\sigma(y) \right)^{1/2} \le$$

$$\le C^{1/2} \left(\int_\Sigma |K(x,y)||f(y)|^2 d\sigma(y) \right)^{1/2}.$$

Squaring both sides and integrating (in x) over Σ leads to

$$\int_\Sigma |Tf(x)|^2 d\sigma(x) \leq C \int_\Sigma \int_\Sigma |K(x,y)||f(y)|^2 \sigma(y) d\sigma(x) =$$

$$= C \int_\Sigma \left[\int_\Sigma |K(x,y)||f(y)|^2 \sigma(x) \right] d\sigma(y) =$$

$$= C \int_\Sigma \left[\int_\Sigma |K(x,y)| d\sigma(x) \right] |f(y)|^2 d\sigma(y) \leq$$

$$\leq C^2 \int_\Sigma |f(y)|^2 d\sigma(y) = C^2 \|f\|^2_{L^2(\Sigma)}.$$

Taking the square root of both sides in the last inequality proves the lemma. □

Theorem 4.2. *The operators T and T^* defined by (31) and (32) are compact.*

Proof: We approximate the operator T by T_ϵ where T_ϵ is defined by

$$T_\epsilon f(x) = \int_{\partial D} K_\epsilon(x,y) f(y) d\sigma(y),$$

and

$$K_\epsilon(x,y) = \psi_\epsilon(x,y) K(x,y)$$

where $\psi_\epsilon \in C^\infty$, $0 \leq \psi_\epsilon \leq 1$, $\psi_\epsilon(x,y) = 1$ when $|x-y| \geq \epsilon$ and $\psi_\epsilon(x,y) = 0$ when $|x-y| \leq \epsilon/2$. Then K_ϵ is a bounded function and therefore $K_\epsilon \in L^2(\Sigma \times \Sigma)$ (here we also use that D is a bounded $C^{1,\alpha}-$domain). It follows from Definition 4.3 that T_ϵ is a Hilbert-Schmidt operator and therefore, by Theorem 4.1. If we can show that $T_\epsilon \to T$ then it follows, from Proposition 4.1, that T is compact.

To see that $T_\epsilon \to T$ we will use Lemma 4.1. Therefore we estimate, for small $\epsilon > 0$,

$$\sup_x \int_{\partial D} |K(x,y) - K_\epsilon(x,y)| \, d\sigma(y) = \sup_x \int_{\partial D} (1 - \psi_\epsilon(x,y)) |K(x,y)| d\sigma(y) \leq$$

$$\leq C \sup_x \int_{B_\epsilon(x) \cap \partial D} \frac{1}{|x-y|^{n-1-\alpha}} d\sigma(y) \leq C\epsilon^\alpha,$$

where the constant C only depend on the dimension and the $C^{1,\alpha}$ properties of ∂D. A similar estimate also holds for $\sup_y$. It follows, from Lemma 4.1, that

$$\|(T - T_\epsilon) f\|_{L^2} \leq C\epsilon^\alpha \|f\|_{L^2},$$

and therefore that $T_\epsilon \to T$. Since each T_ϵ is compact it follows by Proposition 4.1 that T is compact.

That T^* is compact follows from Proposition 3.4 and Proposition 4.2. □

4.1 Riesz-Schauder Theory.

We begin this sub-section with a simple lemma.

Lemma 4.2. *Let $\mathcal{B}$ be a Banach space and $M \subset \mathcal{B}$ be a closed linear subspace (not equal to $\mathcal{B}$). Then for every $t < 1$ there exists an $x_t \in \mathcal{B}$ such that $\|x_t\|_{\mathcal{B}} = 1$ and $dist(x_t, M) \geq t$.*

Proof: Let $x \in \mathcal{B} \setminus M$. Since M is closed there is a well defined distance $\text{dist}(x, M) = d > 0$. Now we may pick an $y_t \in M$ such that

$$\|x - y_t\| \leq \frac{d}{t}. \tag{46}$$

With this choice of y_t the element $x_t = \frac{x-y_t}{\|x-y_t\|}$ satisfies $\|x_t\| = 1$ and for any $y \in M$

$$\|x_t - y\| = \frac{\big\|x_t - y_t - \|y_t - x\|y\big\|}{\|y_t - x\|} \geq \frac{d}{\|y_t - x\|} \geq t,$$

where we used the definition of x_t, then the definition of the distance (and that $y_t - cy \in M$) and finally (46). $\square$

We are now ready to prove the main theorem of this section.

Theorem 4.3. [The Fredholm alternative in Banach Spaces.] *Assume that $T : \mathcal{B} \mapsto \mathcal{B}$ is a compact linear mapping on the Banach space $\mathcal{B}$. Then*

1. *either $Tx = x$ has a nontrivial solution*[14]*; that is the nulls pace $\mathcal{N}(T-I) \neq \{0\}$*
2. *or for each $y \in \mathcal{B}$ the equation*

$$x - Tx = y$$

has a unique solution $x \in \mathcal{B}$ and the operator $(I - T)^{-1} : \mathcal{B} \mapsto \mathcal{B}$ is bounded.

Proof: Since the proof is long we will do it in five shorter steps.

Step 1: *Let $S = I - T$ and $\mathcal{N} = \{x \in \mathcal{B};\ Sx = 0\}$ be the null-space of S. Then there exists aconstant C_0 such that*

$$\text{dist}(x, \mathcal{N}) \leq C_0\|Sx\| \qquad \text{for all } x \in \mathcal{B}.$$

Proof of Step 1: We will argue by contradiction and assume that there is a sequence $x_k \in \mathcal{B}$ such that

$$\text{dist}(x_k, \mathcal{N}) \geq k\|Sx_k\|. \tag{47}$$

Since S is linear there is no loss of generality to assume that $\|Sx_k\| = 1$.

[14] A nontrivial solutions is a solution $x \neq 0$.

We will need to assure that x_k does not converge to infinity in order to use the compactness of T. Therefore we define

$$z_k = \frac{x_k - y_k}{\|x_k - y_k\|},$$

where $y_k \in \mathcal{N}$ is chosen so that

$$\mathrm{dist}(x_k, \mathcal{N}) \leq \|x_k - y_k\| \leq 2\mathrm{dist}(x_k, \mathcal{N}). \tag{48}$$

By construction $\|z_k\| = 1$ and form (47) and linearity of S we may conclude that

$$\|Sz_k\| \leq \frac{1}{k} \to 0. \tag{49}$$

Now we use that T is compact which implies that $Tz_k \to y_0 \in \mathcal{B}$ for some y_0 (at least for a sub-sequence of z_k), it follows from (49) that

$$0 \leftarrow Sz_k = (I - T)z_k = z_k - Tz_k \Rightarrow z_k \to y_0.$$

By (49) and continuity of S it follows that $y_0 \in \mathcal{N}$. But $y_0 \notin \mathcal{N}$ since z_k was constructed so that $\mathrm{dist}(z_k, \mathcal{N}) \geq 1/2$:

$$\mathrm{dist}(z_k, \mathcal{N}) = \inf_{y \in \mathcal{N}} \|z_k - y\| = \inf_{y \in \mathcal{N}} \left\| \frac{x_k - y_k}{\|x_k - y_k\|} - y \right\| =$$

$$= \inf_{y \in \mathcal{N}} \frac{\big\|x_k - y_k - \|x_k - y_k\| y\big\|}{\|x_k - y_k\|} = \frac{\mathrm{dist}(x_k, \mathcal{N})}{\|x_k - y_k\|} \geq \frac{1}{2},$$

where we used (48) in the last inequality. This proves step 1.

Step 2: *The range of S is closed in $\mathcal{B}$.*

Proof of Step 2: We need to show that if $Sx_k \to y_0 \in \mathcal{B}$ then there exists an $x \in \mathcal{B}$ such that

$$Sx = y_0. \tag{50}$$

Using that, by step 1, $\mathrm{dist}(x_k, \mathcal{N}) \leq C_0 \|Sx_k\|$ and that $\|Sx_k\|$ is bounded (since it is convergent) it follows from Step 1 that $\mathrm{dist}(x_k, \mathcal{N})$ is bounded. We may therefore find $y_k \in \mathcal{N}$ so that $z_k = x_k - y_k$ is bounded and

$$Sz_k = Sx_k - Sy_k \to y_0 - 0. \tag{51}$$

Since z_k is bounded and T is compact there exist a sub-sequence, that we may assumt to be the full sequence in order to simplify notation, such that $Tz_k \to z_0$ for some $z_0 \in \mathcal{B}$. Therefore

$$z_k = Iz_k - Tz_k + Tz_k = Sz_k + Tz_k \to y_0 + z_0. \tag{52}$$

Using (52) together with continuity of S it follows that

$$S(y_0 + z_0) = \lim_{k \to \infty} Sz_k = y_0,$$

where we used (51) in the last equality. This proves (50) with $x = y_0 + z_0$ and therefore step 2.

Step 3: *If $Sx = 0$ does not have a non-trivial solution then $Sx = y$ has a unique solution for every $y \in \mathcal{B}$.*

Proof of Step 3: We assume that $Sx = 0$ implies that $x = 0$, that is the null-space $\mathcal{N} = \{0\}$. We will use S^j for the composition of S $j-$times: $S^2(x) = S(Sx)$, $S^3 = S(S(Sx))$ et.c. Then Range(S^j) is a non-increasing sequence of closed (by step 2) sub-spaces of $\mathcal{B}$. We claim that there is a j such that Range(S^j) = Range(S^{j+1}). If not then, by Lemma 4.2, there is a sequence $x_j \in$ Range(S^j) such that $\|x_j\| = 1$ and

$$\text{dist}(x_j, \text{Range}(S^{j+1})) \geq \frac{1}{2}. \tag{53}$$

Therefore, for $k > j$

$$Tx_j - Tx_k = x_j + (-x_k - Sx_j + Sx_k) = x_j - y \tag{54}$$

for some $y \in$ Range(S^{j+1}). By (53) and (54) it follows that

$$\|Tx_j - Tx_k\| \geq \frac{1}{2} \tag{55}$$

contradicting that Tx_j has a convergent sub-sequence and therefore the compactness of T. It follows that Range(S^j) = Range(S^{j+1}) for some j.

Next we use that $\mathcal{N} = \{0\}$ by the assumption in step 3. To use this we let $y \in \mathcal{B}$ be arbitrary and j be so large that Range(S^j) = Range(S^{j+1}). Then $S^j y \in$ Range(S^j) = Range(S^{j+1}) which implies that there is an $x \in \mathcal{B}$ such that $S^{j+1}x = S^j y$. Using linearity of S we may conclude that

$$0 = S^{j+1}x - S^j y = S^j(Sx - y). \tag{56}$$

Since $\mathcal{N} = \{0\}$ equation (56) implies that $Sx - y = 0$ which implies that $Sx = y$. This proves that $Sx = y$ has a solution for every $y \in \mathcal{B}$.

To see that in $\mathcal{N} = \{0\}$ then the solution is unique is trivial. If $Sx_1 = Sx_2 = y$ for some y then $S(x_1 - x_2) = 0$ which implies that $x_1 = x_2$. This proves step 3.

Step 4: *If Range$(S) = \mathcal{B}$ then $Sx = 0$ implies that $x = 0$. In particular case 1 or case 2 holds.*

Proof of Step 4: We will use the notation $\mathcal{N}_j = \{x;\ S^j x = 0\}$ for the null-space of S^j. Then $\mathcal{N}_j \subset \mathcal{N}_{j+1}$ and, since S is continuous, each $\mathcal{N}_j$ is a closed subspace of $\mathcal{B}$. We will argue as in step 3 in order to show that there is a j such that $\mathcal{N}_j = \mathcal{N}_{j+1}$. In particular if no such j exists then we may find a sequence $x_j \in \mathcal{N}_j$ such that $\|x_j\| = 1$ and

$$\text{dist}(x_j, \mathcal{N}_{j-1}) \geq 1/2. \tag{57}$$

It follows that for $l > j$

$$Tx_l - Tx_j = x_l + (-x_j + Sx_j - Sx_l). \tag{58}$$

Since $\text{dist}(x_l, T_{l-1}) \geq 1/2$ and $(-x_j + Sx_j - Sx_l) \in \mathcal{N}_{l-1}$ it follows from (58) that $\|Tx_l - Tx_j\| \geq 1/2$. Therefore Tx_j does not have a convergent sub-sequence contradicting the compactness of T. We may conclude that $\mathcal{N}_{j_0} = \mathcal{N}_{j_0+1}$ for some j_0.

If $\text{Range}(S) = \mathcal{B}$ then for any y there exist an $x \in \mathcal{B}$ such that $S^{j_0}x = y$. If also $y \in \mathcal{N}_{j_0}$ it follows that $S^{2j_0}x = 0$. But this implies that $x \in \mathcal{N}_{2j_0} = \mathcal{N}_{j_0}$ wherefore $0 = S^{j_0}x = y$. We have proved that $y = 0$ for any $y \in \mathcal{N}_{j_0}$. Since $\mathcal{N}_0 \subset \mathcal{N}_1 \subset ... \subset \mathcal{N}_{j_0}$ it follows that $\mathcal{N}_0 = \{0\}$. Step 4 follows.

Step 5: *If we are in case 2, that $Sx = y$ has a unique solution for every $y \in \mathcal{B}$, then S^{-1} exists and is bounded.*

Proof of Step 5: Under the assumptions S is surjective and by step 4 $\mathcal{N} = \{0\}$ and therefore S is injective. It follows that S is a bijection and therefore invertible.

That S^{-1} is bounded follows directly from step 1 since if $\mathcal{N} = \{0\}$ then $\text{dist}(x, \mathcal{N}) = \|x\|$ and we may therefore rewrite the conclusion of step 1 as

$$\|S^{-1}x\| \leq C_0\|x\|.$$

This finishes the proof. $\square$

4.2 Riesz-Schauder Theroey in Hilbert Spaces.

Next we would like to prove the Fredholm alternative in Hilbert spaces. Since every Hilbert space is a Banach space Theorem 4.3 naturally holds in Hilbert spaces. But since Hilbert spaces have more structure than Banach spaces we are able to prove a slightly stronger version in Hilbert spaces. We begin with a lemma.

Lemma 4.3. *Let $T : \mathcal{H} \mapsto \mathcal{H}$ be a bounded linear operator on $\mathcal{H}$. Then the closure of the range of T, $\overline{\text{Range}(T)}$, is the orthogonal complement of the null space of T^*.*

Proof: Let $\mathcal{R} = \text{Range}(T)$ and $\mathcal{N}^*$ be the null-space of T^*. To see that $\mathcal{R}$ is contained in the complement of $\mathcal{N}^*$ we let $y \in \mathcal{R}$ and $z \in \mathcal{N}^*$. Since $y \in \mathcal{R}$ there exists an $x \in \mathcal{H}$ such that $Tx = y$ and therefore

$$(y, z) = (Tx, z) = (x, T^*z) = (x, 0) = 0,$$

therefore y is orthogonal to every element in $\mathcal{N}^*$ and it follows that $\mathcal{R} \subset (\mathcal{N}^*)^\perp$. Since $(\mathcal{N}^*)^\perp$ is closed (Exercise 3) it follows that $\overline{\mathcal{R}} \subset (\mathcal{N}^*)^\perp$.

In order to show that $\overline{\mathcal{R}} = (\mathcal{N}^*)^\perp$ we need to show that $(\mathcal{N}^*)^\perp \subset \overline{\mathcal{R}}$. It is enough to show that if $x \notin \overline{\mathcal{R}}$ then $x \notin (\mathcal{N}^*)^\perp$. To see this we pick an $x \notin \overline{\mathcal{R}}$. We may write $x = x_1 + x_2$ where $x_1 \in \overline{\mathcal{R}}$ and $x_2 \in \overline{\mathcal{R}}^\perp$, where $x_2 \neq 0$ since

$x \notin \overline{\mathcal{R}}$. It is enough to show that $x_2 \notin (\mathcal{N}^*)^\perp$. Since $x_2 \in \overline{\mathcal{R}}^\perp$ it follows that for every $y \in \mathcal{H}$

$$0 = (Ty, x_2) = (y, T^* x_2).$$

But if $(y, T^* x_2) = 0$ for every $y \in \mathcal{H}$ then $T^* x_2 = 0$, which is seen by choosing $y = T^* x_2$, and therefore $x_2 \in \mathcal{N}^*$. We may conclude that $x_2 \notin (\mathcal{N}^*)^\perp$. The lemma follows. □

Theorem 4.4. [The Fredholm alternative in Hilbert Spaces.] *Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \mapsto \mathcal{H}$ be a compact linear operator. Then there exists a countable set $\Lambda \subset \mathbb{R}$ with the following property.*

1) *If $\lambda \notin \Lambda$ and $\lambda \neq 0$ then for every $y \in \mathcal{H}$ the equations*

$$\lambda x - Tx = y \quad \text{and } \lambda x - T^* x = y \tag{59}$$

have a uniquely determined solution $x \in \mathcal{H}$.

2) *If $\lambda \in \Lambda$ then the mappings $\lambda I - T$ and $\lambda I - T^*$ have equal, finite and non-zero dimension of their null-spaces. Furthermore $\lambda x - Tx = y$ is solvable if and only if $y \in Ker(\lambda I - T^*)^\perp$ and $\lambda x - T^* x = y$ is solvable if and only if $y \in Ker(\lambda I - T)^\perp$.*

Proof: We refer to the two cases in Theorem 4.3. If $\lambda x - Tx = 0$ has no non-trivial solutions then case 2 of Theorem 4.3 implies that $\lambda x - Tx = y$ has a unique solution for every $y \in \mathcal{H}$. This implies that the orthogonal complement of $\overline{\text{Range}(\lambda I - T)}$ and, by Lemma 4.3, the null space of $\lambda I - T^*$ must be $\{0\}$.

If $\text{Null}(\lambda I - T^*) = \{0\}$ then, by Theorem 4.3, $\lambda x - T^* x = y$ has a unique solution. This shows that **1)** holds if λ is such that $\lambda x - Tx = 0$ only have the trivial solution. We may define Λ to be the complement of the set of $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\lambda x - Tx = 0$ only have the trivial solution.

If $\lambda \in \Lambda$, that is if $\lambda x - Tx = 0$ has non-trivial solutions, then we need to show that the second case in the theorem is satisfied. First we show that the dimension of $\text{Null}(\lambda I - T)$ is finite.

If $\text{Dim}(\text{Null}(\lambda I - T))) = \infty$ then we let x_j be an orthonormal basis of $\text{Null}(\lambda I - T)$. That is for every $j \in \mathcal{N}$

$$Tx_j = \lambda x_j$$

and thus $\|Tx_j\| = \|\lambda x_j\| = |\lambda| \|x_j\| = |\lambda|$. But since T is compact there should be a convergent sub-sequence $Tx_j = \lambda x_j$ which contradicts the orthonormality of x_j. It follows that the basis x_j of $\text{Ker}(\lambda I - T)$ cannot be infinite. A similar argument shows that the dimension of $\text{Ker}(\lambda I - T^*)$ must be finite.

That $\lambda x - Tx = y$ is solvable if and only if $y \in \text{Ker}(\lambda I - T^*)^\perp$ follows from Lemma 4.3.

Finally we need to verify that $\text{Ker}(\lambda I - T)$ and $\text{Ker}(\lambda I - T^*)$ has equal dimension. To that end we assume that $\text{Dim}(\text{Ker}(\lambda I - T)) = n$ and that $\text{Dim}(\text{Ker}(\lambda I - T^*)) = m$. Let us for definiteness assume that $n \leq m$ (if $m \leq n$ is handled similarly) and show that $n = m$. The strategy to show that $n = m$

will consist in constructing a compact operator A_n with $\mathrm{Dim}(\mathrm{Ker}(A_n)) = 0$ and $\mathrm{Dim}(\mathrm{Ker}(A_n^*)) = n - m$ - it follows from this and part **2)** of Theorem 4.3 that $n = m$.

Let $\{x_1, x_2, ..., x_n\}$ be an orthonormal basis for $\mathrm{Ker}(\lambda I - T)$ and $\{y_1, y_2, ..., y_m\}$ be a basis for $\mathrm{Ker}(\lambda I - T^*)$. Define the operator $A_1 : \mathcal{H} \mapsto \mathcal{H}$

$$A_1 x = \lambda I x - T x - (x_n, x) y_m.$$

We claim that $\mathrm{Dim}(\mathrm{Ker}(A_1) = n - 1$ and that $\mathrm{Dim}(\mathrm{Ker}(A_1^*)) = m - 1$. To see this we pick an $x \in \mathrm{Ker}(A_1)$. Then

$$\left(\frac{1}{2} I - T\right) x = (x_n, x) y_m \in \mathrm{Ker}(\lambda - T^*). \tag{60}$$

From Lemma 4.3 we know that $\overline{\mathrm{Range}(\lambda I - T)} = \mathrm{Ker}(\lambda - T^*)^{\perp}$. This together with (60) implies that $(\lambda I - T)x = (x_n, x)y_m$ is orthogonal to the range of $\lambda I - T$. We may conclude that $(\lambda I - T)x = (x_n, x)y_m = 0$; that is $x \in \mathrm{Ker}(\lambda I - T)$.

Since $\{x_1, ..., x_n\}$ is a basis for $\mathrm{Ker}(\lambda - T)$ it follows that we may write, for some $a_1, ..., a_n \in \mathbb{R}$,

$$x = \sum_{j=1}^{n} a_j x_j.$$

But we have already shown that $(x_n, x) = a_n = 0$ so any $x \in \mathrm{Ker}(A_1)$ can be written

$$x = \sum_{j=1}^{n-1} a_j x_j.$$

It follows that $\mathrm{Dim}(\mathrm{Ker}(A_1)) \leq n - 1$. That $\mathrm{Dim}(\mathrm{Ker}(A_1)) \geq n - 1$ follows directly from $x_1, x_2, ..., x_{n-1} \in \mathrm{Ker}(A_1))$. We have thus shown that $\mathrm{Dim}(\mathrm{Ker}(A_1)) = n - 1$.

To see that $\mathrm{Dim}(\mathrm{Ker}(A_1^*)) = m - 1$ we argue similarly. Pick a $y \in \mathrm{Ker}(A_1^*)$. We aim to show that $y \in \mathrm{Ker}(\lambda I - T^*)$ in order to do that we need to calculate the dual of the operator $S : \mathcal{H} \mapsto \mathcal{H}$ defined by $S(x) = (x_n, x)y_m$. The dual is defined by

$$(Sx, z) = (x_n, x)(y_m, z) = (x, S^* z).$$

We can conclude that $S^* z = (y_m, z)x_n$. Since $y \in \mathrm{Ker}(A_1^*)$ it follows that

$$A_1^* y = (\lambda - T^*)y - S^* y = 0 \Rightarrow (\lambda I - T^*)y = (y_m, y)x_n.$$

We may again refer to Lemma 4.3 to conclude that $(y_m, y)x_n \in \mathrm{Range}(\lambda I - T^*)^{\perp}$, thus $(y_m, y) = 0$ and $y \in \mathrm{Ker}(\lambda I - T^*)$. We may therefore write

$$y = \sum_{j=1}^{m-1} b_j y_j$$

and it follows that $\mathrm{Ker}(A_1^*)$ is $m - 1$ dimensional.

Inductively we may define the operators $A_2, A_3, ..., A_n$ such that $\mathrm{Dim}(\mathrm{Ker}(A_k)) = n-k$ and $\mathrm{Dim}(\mathrm{Ker}(A_k^*)) = m-k$. But this implies that $\mathrm{Dim}(\mathrm{Ker}(A_n)) = 0$ and therefore, by part **1)** of this theorem both $A_n x = y$ and $A_n^* x = y$ have unique solutions for every $y \in \mathcal{H}$. But this implies that $0 = \mathrm{Dim}(\mathrm{Ker}(A_n^*)) = m - n$. We have thus shown that $n = m$. $\square$

4.3 Exercises

1. Formulate and prove the Fredholm alternative in the Hilbert space $\mathbb{R}^n$.
2. Let D be a bounded set.
   (a) Show that $C^\infty(D)$ is a linear subspace in $L^2(D)$.
   (b) Show that $C^\infty(D) \neq L^2(D)$.
   It is true (try to prove it!) that $\overline{C^\infty(D)} = L^2(D)$, this shows that we may have a proper subspace of an infinite dimensional space whose closure is the entire space.
   (c) Is the same true in $\mathbb{R}^n$. That is, does it exist a proper linear subspace $S \subset \mathbb{R}^n$ such that $\overline{S} = \mathbb{R}^n$. Prove your result.
3. Let $S : \mathcal{B} \mapsto \mathcal{B}$ be a linear and continuous operator on the Banach space $\mathcal{B}$.
   a) Show that
   $$\mathrm{Ker}(S) = \{x \in \mathcal{B};\ Sx = 0\}$$
   is a closed subspace.
   b) If $\mathcal{B}$ is a Hilbert space show that the orthogonal complement of $\mathrm{Ker}(S)$ is also closed.
4. In the proof of Proposition 4.1 we showed that we can, for every $\epsilon > 0$, find a subsequence x_{l_j} such that $\|Tx_{l_j} - Tx_{l_m}\| < \epsilon$ whenever $j, m > J_\epsilon$.
   (a) Show, maybe by means of an example, that this does not imply that Tx_{l_j} converges.
   (b) Show that by a standard diagonalization argument (as in the proof of the Arzela-Ascoli Theorem) one may save the proof.[15]

5 Existence of solutions to the Dirichlet and Neumann problems.

In this section we will show the existence of solutions to the Dirichlet and Neumann problems in bounded $C^{1,\alpha}-$domains. But before we do that we need

[15] It is rather usual in long and complicated texts to exclude standard arguments if it can be assumed that the reader can fill in the details. It is arguable whether the proof of Proposition 4.1 is complete, it depends on what the reader think is obvious.

to resolve a slight inconsistency in the theory we have developed so far. It has been important to work in a Hilbert space $L^2(\partial D)$; but our initial goal was to show that $\frac{1}{2}I + T : C(\partial D) \mapsto C(\partial D)$ was onto. Also, to assure that u has the right boundary data we need to show that u is continuous in $\overline{D}$ - for that we need $f \in C(\partial D)$.

Clearly if we show that we can always find solutions for $f \in L^2(\partial D)$ then we can find solutions for every $f \in C(\partial D)$[16] But we will in any case show that the operator $\frac{1}{2}I - T : C(\partial D) \mapsto C(\partial D)$ is onto.

Lemma 5.1. *Assume that* $|K(x,y)| \leq \frac{C_K}{|x-y|^{n-1-\alpha}}$ *for some constant* C_K *and* $\alpha > 0$ *and that* $K(x,y)$ *is continuous when* $x \neq y$*. Furthermore let* D *be a bounded* $C^{1,\alpha}-$*domain in* $\mathbb{R}^n$*. Then the operator*

$$T_K \phi(x) = \int_{\partial D} K(x,y)\phi(y) d\sigma(y)$$

maps bounded functions to continuous functions.

Proof: Let x and y be arbitrary points and $\epsilon > 0$ given. We choose a small $\delta > 0$ and, and points x and y such that $|x-y| < \delta$, calculate

$$|T_K f(x) - T_K f(y)| = \left| \int_{\partial D} \left(K(x,z) - K(y,z)\right) f(z) d\sigma(z) \right| \leq$$

$$\leq \left| \int_{\partial D \cap \{|x-z|<2\delta\}} \left(K(x,z) - K(y,z)\right) f(z) d\sigma(z) \right| + \tag{61}$$

$$+ \left| \int_{\partial D \setminus \{|x-z|<2\delta\}} \left(K(x,z) - K(y,z)\right) f(z) d\sigma(z) \right| =$$

$$= I_1 + I_2.$$

Since f is bounded we may estimate, since $|x-y| < \delta$,

$$I_1 \leq \sup_{x \in \partial D} |f(x)| \int_{\partial D \cap \{|x-z|<2\delta\}} \left(\frac{C_K}{|x-z|^{n-1-\alpha}} + \frac{C_K}{|y-z|^{n-1-\alpha}} \right) d\sigma(z) \leq$$

$$\leq C_\alpha C_K \delta^{1-\alpha} \sup_{x \in \partial D} |f(x)|,$$

where the last estimate follows from integrating in polar coordinates and the constant C_α depends on the $C^{1,\alpha}$ character of Ω as well as on α. This implies that if $|x-y|$ is small enough, so that we may choose δ small enough, then $I_1 < \epsilon/2$.

To see that also $I_2 < \epsilon/2$ we just that for $z \in \partial D \setminus B_{2\delta}(x)$ the kernel $K(y,z)$ is continuous in y for $y \in \partial D \cap B_\delta(x)$. Since D is bounded it follows that $K(y,z)$ is uniformly continuous on $z \in \partial D \setminus B_{2\delta}(x)$ and $y \in D \cap B_\delta(x)$ and therefore

[16] Since $C(\partial D) \subset L^2(\partial D)$.

that $K(y,z) \to K(x,z)$ uniformly in z. It follows that $I_2 < \epsilon/2$ if $|x-y|$ is small enough.

Using the estimates on I_1 and I_2 together with (61) implies that for every $\epsilon > 0$ it follows that if $|x-y|$ is small enough then

$$|T_K f(x) - T_K f(y)| < \epsilon.$$

$\square$

Proposition 5.1. *If $f \in C(\partial D)$ and $\frac{1}{2}\phi(x) \pm T\phi(x) = f(x)$ then $\phi \in C(\partial D)$. A similar result holds for $\frac{1}{2}I \pm T^*$.*

Proof: We will only prove the proposition for $\frac{1}{2}\phi(x) - T\phi(x) = f(x)$, the other cases are proven in exactly the same way.

The idea of the proof is to approximate the kernel $K(x,y) = \frac{\partial N(x,y)}{\partial \nu_y}$ by $K_\epsilon(x,y) = \psi_\epsilon(x,y)K(x,y)$ where $\psi_\epsilon \in C^\infty(\mathbb{R}^{2n})$ and $\psi_\epsilon(x,y) = 1$ when $|x-y| \geq \epsilon$ and $\psi_\epsilon(x,y) = 0$ when $|x-y| \leq \epsilon/2$. Then the operator T_ϵ, where T_ϵ is the integral operator with kernel K_ϵ, maps $L^2(\partial D)$ functions to continuous functions since

$$|T_\epsilon\phi(x) - T_\epsilon\phi(y)| = \left| \int_{\partial D} \left(K_\epsilon(x,z)\phi(z) - K_\epsilon(y,z)\phi(z)\right) d\sigma(z) \right| \leq$$

$$\leq \|\phi\|_{L^2(\partial D)} \left(\int_{\partial D} |K_\epsilon(x,z) - K_\epsilon(y,z)|^2 d\sigma(z) \right)^{1/2}. \tag{62}$$

But since $K_\epsilon(x,y)$ is continuous the right side of (62) will converge to zero as $x \to y$. It follows that $T_\epsilon\phi$ is continuous.

Next we notice that

$$T\phi(x) = T_\epsilon\phi(x) - \underbrace{(T - T_\epsilon)}_{=S_\epsilon}\phi(x),$$

where we define $S_\epsilon : L^2(\partial D) \mapsto L^2(\partial D)$ by the above expression. Next we notice that we may also consider $S_\epsilon : L^\infty(\partial D) \mapsto L^\infty(\partial D)$. We claim that the operator norm $\|S_\epsilon\|_{L^\infty \mapsto L^\infty}$ is bounded by $C\epsilon^\alpha$ where the constant C is independent of ϵ. To see this we calculate

$$\|S_\epsilon\phi\|_{L^\infty(\partial D)} = \left\| \int_{\partial D} K(x,y)\left(1 - \psi_\epsilon(x,y)\right)\phi(y) d\sigma(y) \right\|_{L^\infty(\partial D)} \leq$$

$$\leq \|\phi\|_{L^\infty(\partial D)} \sup_{x \in \partial D} \left| \int_{\partial D} K(x,y)\left(1 - \psi_\epsilon(x,y)\right) d\sigma(y) \right| \leq$$

$$\leq \|\phi\|_{L^\infty(\partial D)} \sup_{x \in \partial D} \int_{B_\epsilon(x) \cap \partial D} |K(x,y)| d\sigma(y) \leq$$

$$\leq C\|\phi\|_{L^\infty(\partial D)} \int_{B_\epsilon(x) \cap \partial D} \frac{1}{|x-y|^{n-1-\alpha}} d\sigma(y) \leq C\epsilon^\alpha \|\phi\|_{L^\infty(\partial D)}.$$

It follows that the operator norm $\|S_\epsilon\|_{L^\infty(\partial D)\mapsto L^\infty(\partial D)} \leq C\epsilon^\alpha$. In particular, we may chose ϵ so small that $\|S_\epsilon\|_{L^\infty(\partial D)\mapsto L^\infty(\partial D)} \leq \frac{1}{4}$.

From Proposition 3.3 we may conclude that $\frac{1}{2}I - S_\epsilon$ is invertable, with inverse $\sum_{k=0}^\infty 2^{k-1}S_\epsilon^k$.

Now

$$\left(\frac{1}{2}I - T\right)\phi(x) = f(x) \in C(\partial D) \tag{63}$$

is equivalent to

$$\left(\frac{1}{2}I - S_\epsilon\right)\phi(x) = f(x) - T_\epsilon\phi(x) \in C(\partial D). \tag{64}$$

By (64) and Proposition 3.3 it follows that

$$\phi(x) = \left(\frac{1}{2}I - S_\epsilon\right)^{-1}(f(x) - T_\epsilon\phi(x)) = \sum_{k=0}^\infty 2^{k-1}S_\epsilon^k(f(x) - T_\epsilon\phi(x)). \tag{65}$$

The right side of (65) converges uniformly (since $\|S_\epsilon\|_{L^\infty(\partial D)\mapsto L^\infty(\partial D)} \leq \frac{1}{4}$) and by Lemma 5.1 each of the functions $S_\epsilon^k(f - T_\epsilon\phi)$ is continuous. It follows that $\phi(x)$ is continuous. □

Corollary 5.1. *If $\frac{1}{2}I \pm T : L^2(\partial D) \mapsto L^2(\partial D)$ is surjective then $\frac{1}{2}I \pm T : C(\partial D) \mapsto C(\partial D)$ is also surjective. (Similarly holds for $\frac{1}{2}I \pm T^*$.)*

Proof: Since for any function $f \in C(\partial D) \subset L^2(\partial D)$ we may find a solution $\phi \in L^2(\partial D)$ such that $\frac{1}{2}\phi(x) - T\phi(x) = f(x)$ (we work with minus sign for definiteness, the proof for plus sign is similar). But Proposition 5.1 then implies that $\phi \in C(\partial D)$. That is, for every $\in C(\partial D)$ there is an $\phi \in C(\partial D)$ such that $\frac{1}{2}\phi(x) - T\phi(x) = f(x)$. It follows that $\frac{1}{2}I - T : C(\partial D) \mapsto C(\partial D)$ is onto. □

Next we need to investigate the kernels of T and T^* in order to use Riesz-Schauder Theorem.

Proposition 5.2. Part 1: *Assume that D is a bounded connected $C^{1,\alpha}-$domain then the kernels of $\frac{1}{2}I - T$ and $\frac{1}{2}I - T^*$, defined in (31) and (32), have dimension 1.*

Part 2: *Furthermore, if D^c has m bounded components then the kernels of $\frac{1}{2}I + T$ and $\frac{1}{2}I + T^*$ have dimension m.*

Proof: Again we have two different, but similar, cases to prove. We will only provide details for Part 1 of the proposition. At the end of the proof we will indicate the difference in the proof of Part 2 of the theorem. But we will leave the details in the proof of Part 2 as an exercise (a rather difficult exercise).

We begin by claiming that all constants $c \in \mathbb{R}$ are contained in the kernel of $\frac{1}{2}I - T$. This follows from Lemma 2.1 since if $u(y)$ is defined by

$$u(y) = N(x_0, y)$$

then $u(y)$ is harmonic in D for $x_0 \in \partial D$. In particular,

$$0 = \int_{D \setminus B_\epsilon(x_0)} \Delta_y N(x_0, y) dy = \tag{66}$$

$$= \int_{\partial D \setminus B_\epsilon(x_0)} \frac{\partial N(x_0, y)}{\partial \nu_y} d\sigma(y) + \int_{\partial B_\epsilon(x_0) \cap D} \frac{\partial N(x_0, y)}{\partial \nu_y} d\sigma(y).$$

But as $\epsilon \to 0^+$ the solid angle of $\partial B_\epsilon(x_0) \cap D$ measured from x_0 will converge to half the unit sphere which implies, by Lemma 2.1, that the second integral converges to $-1/2$. Therefore

$$\int_{\partial D} \frac{\partial N(x_0, y)}{\partial y} d\sigma(y) = \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(x_0) \cap D} \frac{\partial N(x_0, y)}{\partial \nu_y} d\sigma(y) = \frac{1}{2}.$$

It follows that, for any constant c,

$$\left(\frac{1}{2} I - T \right) c = \left(\frac{1}{2} - \frac{1}{2} \right) c = 0 \quad \text{ on } \partial D. \tag{67}$$

Therefore all constants are in the kernel of $\frac{1}{2}I - T$. We may conclude that $\text{Dim}(\text{Kernel}(\frac{1}{2}I - T)) \geq 1$.

From Theorem 4.4 we know that the dimension of the kernels of $\frac{1}{2}I - T$ and $\frac{1}{2}I - T^*$ have the same dimension. So to show that the kernels are one dimensional it is enough to show that the kernel of $\frac{1}{2}I - T^*$ has dimension less than or equal to one. To that end we let $f \in \text{Ker}(\frac{1}{2}I - T^*)$ and define

$$u(x) = \int_{\partial D} f(y) N(x, y) d\sigma(y). \tag{68}$$

Then, by Theorem 2.2 and since $f \in \text{Ker}(\frac{1}{2}I - T^*)$, $\frac{\partial u(x)}{\partial \nu_x} = 0$. We may conclude that

$$0 = \int_{\partial D} u(x) \frac{\partial u(x)}{\partial \nu_x} d\sigma(x) = \int_D |\nabla u|^2 dx.$$

Thus u is constant in D.

To show that $\text{Dim}(\text{Ker}(\frac{1}{2}I - T^*)) \leq 1$ it is therefore enough to show that if we have two functions $f_1, f_2 \in \text{Ker}(\frac{1}{2}I - T^*)$ and let u_1 and u_2 be defined by (68) (with f_1 and f_2 in place of f) and $u_1 = u_2 = \textit{constant}$ then $f_1 = f_2$. That is the same as showing that if u defined by (68) is identically zero in D then $f = 0$. Now if $u = 0$ in D then by continuity of u, Lemma 5.1, it follows that u solves the Dirichlet problem in D^c with zero boundary data. Also $\lim_{x \to \infty} u(x) = 0$ which can be seen from the definition of u in (68).[17] It follows from the maximum principle that $u = 0$ in D^c, since u is continuous and zero in D it follows that $u = 0$ in $\mathbb{R}^n$. We may conclude from Corollary 2.2 that

$$f(x) = \frac{\partial u}{\partial \nu_+} - \frac{\partial u}{\partial \nu_-} = 0.$$

[17] This is really the only time we use that $n > 2$ in a fundamental way in these notes.

This finishes the proof of Part 1.

The argument for Part 2 is similar. We begin by writing $D^c = \bigcup_{k=0}^m D_k^c$ where $D_1^c, D_2^c, ..., D_m^c$ are the bounded components of D^c and D_0^c is the unbounded component. Arguing as in (67) we may conclude that the function

$$f_k(x) = \begin{cases} 1 & \text{if } x \in \partial D_k^c \\ 0 & \text{else} \end{cases}$$

is in the kernel of $\frac{1}{2}I + T$. That is $\mathrm{Dim}(\mathrm{Ker}(\frac{1}{2}I + T)) \geq m$.

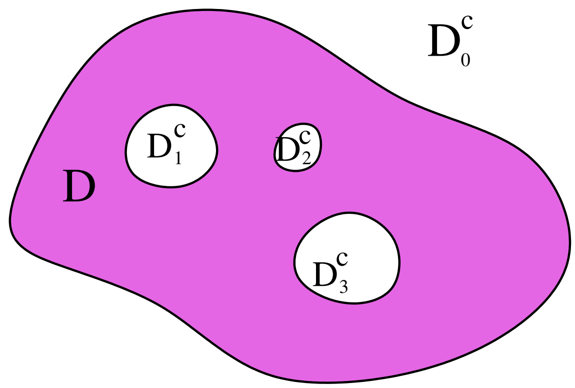

Figure 1: *The domain D is the purple domain and D^c consists of the four parts $D_0^c, ..., D_3^c$. To show that $\mathrm{Dim}(\mathrm{Ker}(\frac{1}{2}I + T)) \geq m$ we may apply the argument for (67) in each of the domains D_k^c, $k = 1, 2, 3$.*

To show that $\mathrm{Dim}(\mathrm{Ker}(\frac{1}{2}I + T)) \leq m$ we use Theorem 4.4 as before to conclude that $\mathrm{Dim}(\mathrm{Ker}(\frac{1}{2}I + T)) = \mathrm{Dim}(\mathrm{Ker}(\frac{1}{2}I + T^*))$. It is therefore enough to show that $\mathrm{Dim}(\mathrm{Ker}(\frac{1}{2}I + T^*)) \leq m$. To that end we choose $f \in \mathrm{Ker}(\frac{1}{2}I + T^*)$ and want to show that $f = \textit{constant}$ on each ∂D_k^c, that is f is contained in the span of $f_1, f_2, .., f_m$. The argument for this is analogous with the argument in part 1. $\square$

Proposition 5.3. Part 1: *Assume that D is a bounded connected $C^{1,\alpha}-$domain then*

$$L^2(\partial D) = Ker(\frac{1}{2}I - T)^{\perp} \oplus Ker(\frac{1}{2}I - T^*)$$

and

$$L^2(\partial D) = Ker(\frac{1}{2}I - T^*)^{\perp} \oplus Ker(\frac{1}{2}I - T).$$

Part 2: *Assume that D is a bounded connected $C^{1,\alpha}-$domain then*

$$L^2(\partial D) = Ker(\frac{1}{2}I + T)^{\perp} \oplus Ker(\frac{1}{2}I + T^*)$$

and

$$L^2(\partial D) = Ker(\frac{1}{2}I + T^*)^{\perp} \oplus Ker(\frac{1}{2}I + T).$$

Proof: The proofs of Part 1 and Part 2 are similar so we will only prove Part 1.

Since both $\mathrm{Ker}(\frac{1}{2}I - T)$ and $\mathrm{Ker}(\frac{1}{2}I - T^*)$ are one dimensional it is enough to show that

$$\mathrm{Ker}(\frac{1}{2}I - T)^\perp \cap \mathrm{Ker}(\frac{1}{2}I - T^*) = \{0\}.$$

If $\phi \in \mathrm{Ker}(\frac{1}{2}I - T)^\perp \cap \mathrm{Ker}(\frac{1}{2}I - T^*)$ then $\frac{1}{2}\phi = T^*\phi$ and, by Theorem 4.4, there exists a ψ such that $\frac{1}{2}\psi - T^*\psi = \phi$.

If we define

$$u(x) = \int_{\partial D} \phi(y) N(x,y) d\sigma(y)$$

and

$$v(x) = \int_{\partial D} \psi(y) N(x,y) d\sigma(y)$$

then $\Delta u = \Delta v = 0$ in D and, by Theorem 2.2, $\frac{\partial u}{\partial \nu_x} = 0$ and $\frac{\partial v}{\partial \nu_x} = \phi$ on ∂D. Since $\Delta u = \Delta v = 0$ in D it follows that

$$0 = \int_D u\Delta v - v\Delta u dx = \int_{\partial D} u\frac{\partial v}{\partial \nu_x} - v\frac{\partial u}{\partial \nu_x} d\sigma(x) = \int_{\partial D} u\frac{\partial v}{\partial \nu_x} d\sigma(x), \tag{69}$$

where we also used that $\frac{\partial u}{\partial \nu_x} = 0$ on ∂D since $\phi \in \mathrm{Ker}(\frac{1}{2}I - T^*)$.

Using that $\frac{\partial v}{\partial \nu_x} = \phi$ and $\frac{1}{2}\phi - T^*\phi = 0$ on ∂D it follows that

$$\frac{\partial v}{\partial \nu_x} = \phi = \phi - \left(\frac{1}{2}\phi - T^*\phi\right) = \frac{1}{2}\phi + T^*\phi. \tag{70}$$

From (70) and Corollary 2.1 it follows that $u\lfloor_{\overline{D^c}}$ has Neumann data ϕ on ∂D^c

$$\frac{\partial u}{\partial \nu} = \phi \quad \text{on } \partial \overline{D}^c. \tag{71}$$

Using (69) we may also conclude that

$$\int_{D^c} |\nabla u(x)|^2 dx = -\int_{\partial D^c} u\frac{\partial u}{\partial \nu_x} d\sigma(x) = 0,$$

it follows that u is constant in D^c. But if u is constant then $0 = \frac{\partial u}{\partial \nu_x} = \phi$, where we also used (71). We have therefore shown that if $\phi \in \mathrm{Ker}(\frac{1}{2}I - T)^\perp \cap \mathrm{Ker}(\frac{1}{2}I - T^*)$ then $\phi = 0$. The proposition follows. $\square$

Corollary 5.2. Part 1: *Under the assumptions of Proposition 5.3 it follows that*

$$L^2(\partial D) = Ker(\frac{1}{2}I - T) \oplus Range(\frac{1}{2}I - T)$$

and

$$L^2(\partial D) = Ker(\frac{1}{2}I - T^*) \oplus Range(\frac{1}{2}I - T^*).$$

Part 2: *Under the assumptions of Proposition 5.3 it follows that*

$$L^2(\partial D) = Ker(\frac{1}{2}I + T) \oplus Range(\frac{1}{2}I + T)$$

and

$$L^2(\partial D) = Ker(\frac{1}{2}I + T^*) \oplus Range(\frac{1}{2}I + T^*).$$

Proof: Since, by part 2 of Theorem 4.4, Range$(\frac{1}{2}I - T) =$ Ker$(\frac{1}{2}I - T^*)^\perp$ and Range$(\frac{1}{2}I - T^*) =$ Ker$(\frac{1}{2}I - T)^\perp$ this is a direct consequence of the previous Proposition. □

We are now ready to prove the main theorem of these notes.

Theorem 5.1. *Assume that D is a bounded connected $C^{1,\alpha}-$domain then*

1. **Exterior Dirichlet Problem.** *For every $f \in C(\partial D^c)$ there exist a solution to the Dirichlet problem*

$$\begin{array}{ll} \Delta u = 0 & \text{in } D^c \\ u = f & \text{on } \partial D^c. \end{array}$$

2. **Interior Neumann problem** *For every $f \in C(\partial D)$ such that*

$$\int_{\partial D} f(x) d\sigma(x) = 0$$

there exist a solution to the Neumann problem

$$\begin{array}{ll} \Delta u = 0 & \text{in } D \\ \frac{\partial u}{\partial \nu_x} = f & \text{on } \partial D. \end{array}$$

3. **Interior Dirichlet problem** *For every $f \in C(\partial D)$ there exist a solution to the Dirichlet problem*

$$\begin{array}{ll} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D. \end{array}$$

4. **Exterior Neumann problem** *If D^c consists of $m+1$ components, $D^c = \bigcup_{k=0}^m D_k^c$, where D_0^c is the unbounded component. Then for every $f \in C(\partial D^c)$ such that, for every $k = 1, 2, ..., m$*

$$\int_{\partial D_k^c} f(x) d\sigma(x) = 0$$

there exist a solution to the Neumann problem

$$\begin{array}{ll} \Delta u = 0 & \text{in } D \\ \frac{\partial u}{\partial \nu_x} = f & \text{on } \partial D. \end{array}$$

Proof: Let us start by proving the theorem for the interior Neumann problem. By Corollary 5.2 and Proposition 5.2 it follows that the co-dimension of Range($\frac{1}{2}I - T^*$) is 1. Also if $f \in$ Range($\frac{1}{2}I - T^*$) then there exist a harmonic function u with Neumann data f and therefore

$$0 = \int_D \Delta u(x)dx = \int_{\partial D} \frac{\partial u}{\partial \nu} d\sigma(x),$$

that is Range($\frac{1}{2}I - T^*$) is orthogonal to the constant functions. The second part of the theorem follows.

To prove the theorem for the exterior Dirichlet problem we also use Corollary 5.2. In particular,

$$L^2(\partial D) = \mathrm{Ker}(\frac{1}{2}I - T) \oplus \mathrm{Range}(\frac{1}{2}I - T).$$

But Ker($\frac{1}{2}I - T$) is one dimensional and, by (67), contains the constants. We may therefore write any function $f \in L^2(\partial D)$ as

$$f(x) = c + \tilde{f}(x),$$

for some constant c and $\tilde{f} \in$ Range($\frac{1}{2} - T$). In particular there exist a harmonic $\tilde{u}$ satisfying $\tilde{u} = \tilde{f}$ on ∂D. It follows that $u(x) = c + \tilde{u}$ solves the Dirichlet problem with boundary data f.

The proofs of parts 3 and 4 are similar. $\square$

5.1 Exercises

1. Use Lemma 5.1 to prove that if $f \in C^2(D)$ then there exists a function $u \in C(\mathbb{R}^n)$ such that

$$\Delta u(x) = f(x) \qquad \text{in } D.$$

2. Let D be a bounded $C^{1,\alpha}-$domain, $f \in C^2(D)$ and $g \in C(\partial D)$. Prove that the following problem has a unique solution

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } D \\ u(x) = g(x) & \text{on } \partial D. \end{array}$$

3. In Theorem 5.1 we assume, for the interior Neumann case, that

$$\int_{\partial D} f(x)d\sigma(x) = 0$$

and similarly for the exterior Neumann case. One might ask if that is due to a fault in the method (and another method would prove the theorem without these assumptions) or if the assumptions are necessary. Prove that Theorem 5.1 is false without these assumptions and therefore the assumptions are necessary.

A The Arzela-Ascoli Theorem.

We know that all continuous functions defined on a set $\mathcal{D} \subset \mathbb{R}^n$ is a vector space. We will denote this vector space by $C^0(\mathcal{D})$. We may also define a metric on $C^0(\mathcal{D})$ in order to make $C^0(\mathcal{D})$ into a metric space. Often when one has a vector space one defines a norm on the space. A norm is a slightly stronger structure on the vector space than a metric. Also, a norm interacts well with multiplication by scalars.

Definition A.1. *Let V be a vector space over some number field K (most often $K = \mathbb{R}$ or $K = \mathbb{C}$). Then we say that a function $\|\cdot\| : V \mapsto \mathbb{R}$ is a norm on V if*

1. *$\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$.*
2. *$\|\lambda v\| = |\lambda|\|v\|$ for all $v \in V$ and all $\lambda \in K$.*
3. *$\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in w$.*

It is easy to see that if $\|v\|$ is a norm on V then $d(v, w) = \|v - w\|$ is a metric on V.

We have total freedom to define any norm we would like on the space $C^0(\mathcal{D})$. However one often feels that a norm is natural if it makes the space into a space with good properties. For instance, we would like to define a norm $\|\cdot\|$ on $C^0(\mathcal{D})$ that makes $C^0(\mathcal{D})$ into a complete space. For that we will use the norm

$$\|f\|_{C^0(\mathcal{D})} = \sup_{x \in \mathcal{D}} |f(x)|. \tag{72}$$

Let us first show that with this definition of norm $C^0(\mathcal{D})$ becommes a complete space.

Proposition A.1. *Let $f_j \in C^0(\mathcal{D})$ be a Cauchy sequence in the norm of (72). That is, for every $\epsilon > 0$ there exist an $N_\epsilon > 0$ such that*

$$j, k > N_\epsilon \Rightarrow \|f_j - f_k\|_{C^0(\mathcal{D})} < \epsilon.$$

Then there exists an $f_0 \in C^0(\mathcal{D})$ such that $\lim_{j \to \infty} \|f_j - f_0\|_{C^0(\mathcal{D})} = 0$. That is: $C^0(\mathcal{D})$ is a complete space with the norm $\|f\|_{C^0(\mathcal{D})}$ defined in (72).

Proof: For any fixed $x^0 \in \mathcal{D}$ we clearly have that

$$|f_j(x^0) - f_k(x^0)| \leq \sup_{x \in \mathcal{D}} |f_j(x) - f_k(x)| < \epsilon,$$

if $j, k > N_\epsilon$. It follows that for every $x^0 \in \mathcal{D}$ the sequence of real numbers $f_j(x^0)$ forms a Cauchy sequence and is therefore convergent. We may therefore define a function $f_0(x) = \lim_{j \to \infty} f_j(x)$.

Also, if $j > N_\epsilon$ then

$$\sup_{x \in \mathcal{D}} |f_j(x) - f_0(x)| = \sup_{x \in \mathcal{D}} \left(\lim_{k \to \infty} |f_j(x) - f_k(x)| \right) < \epsilon. \tag{73}$$

It follows that $f_j \to f_0$ uniformly in x.

We need to show that $f_0 \in C^0(\mathcal{D})$, that is that $f_0(x)$ is continuous. To that end we fix an $x^0 \in \mathcal{D}$, an $\epsilon > 0$ and a $j > N_{\epsilon/3}$ then by (73)

$$\sup_{x\in\mathcal{D}} |f_j(x) - f_0(x)| < \frac{\epsilon}{3}. \tag{74}$$

Also since $f_j \in C^0(\mathcal{D})$ there exists a $\delta > 0$ such that

$$|x - x^0| < \delta \Rightarrow |f_j(x) - f_j(x^0)| < \frac{\epsilon}{3}. \tag{75}$$

It follows from (74) and (75) that if $|x - x^0| < \delta$ then

$$|f_0(x) - f_0(x^0)| = |f_0(x) - f_j(x) + f_j(x) - f_j(x^0) + f_j(x^0) - f_0(x^0)| \leq$$

$$\leq |f_0(x) - f_j(x)| + |f_j(x) - f_j(x^0)| + |f_j(x^0) - f_0(x^0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

It follows that f_0 is continuous in every point $x^0 \in \mathcal{D}$. $\square$

This shows that $C^0(\mathcal{D})$ is a complete space. But will $C^0(\mathcal{D})$ even satisfy the Bolzano-Weierstrass property that every sequence $f_j \in C^0(\mathcal{D})$ that is bounded ($\|f_j\|_{C^0(\mathcal{D})} \leq M$) has a convergent subsequence? The answer is no.

Example: *Let $f_j \in C^0([-1,1])$ be defined by*

$$f_j(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0 \\ jx & \text{if } 0 < x < 1/j \\ 1 & \text{if } 1/j \leq x \leq 1. \end{cases}$$

Clearly $\|f_j\|_{C^0([-1,1])} \leq 1$ so f_j forms a bounded sequence. Also $f_j(x) \to f_0(x)$ for every $x \in [-1,1]$ - but not in the sense that $1 = \|f_j - f_0\|_{C^0([-1,1])} \to 0$). It follows that f_j cannot have any sub-sequence that converges in $C^0([-1,1])$.

We need some other condition on a sequence $f_j \in C^0(\mathcal{D})$ in order to assure that it has a convergent subsequence. The right concept is equicontinuity.

Definition A.2. *Let $\mathcal{F}$ be a set of functions defined in $\mathcal{D}$. We say that $\mathcal{F}$ is equicontinuous at $x \in \mathcal{D}$ if for every $\epsilon > 0$ there exist an $\delta_{x,\epsilon} > 0$ such that*

$$|f(x) - f(y)| \leq \epsilon$$

for all $y \in \mathcal{D}$ such that $|x - y| < \delta_{x,\epsilon}$ and all $f \in \mathcal{F}$.

We also say that $\mathcal{F}$ is equicontinuous in $\mathcal{D}$ if $\mathcal{F}$ is equicontinuous at every $x \in \mathcal{D}$.

Naturally, we may consider a sequence of functions $\{f_j\}_{j=1}^{\infty}$ defined on $\mathcal{D}$ as a set $\mathcal{F} = \{f_j;\ j \in \mathbb{N}\}$ and we may therefore say that a sequence $\{f_j\}_{j=1}^{\infty}$ is equicontinuous at x or in $\mathcal{D}$.

We are now ready to prove a very important theorem. The theorem exploits the fact that the uncountable set $\mathbb{R}$ contains a dense and countable subset $\mathbb{Q}$. The idea has been very important in may areas of mathematics.

Theorem A.1. [ARZELA-ASCOLI'S THEOREM] *Let $\{f_j\}_{j=1}^{\infty}$ be a uniformly bounded sequence of functions defined on $\mathcal{D}$, that is $\sup_{x\in\mathcal{D}}|f_j(x)| \leq C$ for some C independent of j. Assume furthermore that $\{f_j\}_{j=1}^{\infty}$ is equicontinuous in $\mathcal{D}$. Then there exist a sub-sequence $\{f_{j_k}\}_{k=1}^{\infty}$ such that $f_{j_k}(x)$ converges pointwise.*

If we define $f_0(x) = \lim_{k\to\infty} f_{j_k}(x)$ then $f_{j_k} \to f_0$ uniformly on compact subsets and $f_0 \in C(\mathcal{D})$.

Proof: The proof is rather long so we will divide it into several steps.

Step 1: *There is a sub-sequence $\{f_{j_k}\}_{k=1}^{\infty}$ that converges pointwise on a countable dense set of $\mathcal{D}$.*

Consider the intersection of $\mathcal{D}$ and the points with rational coordinates $\mathcal{D}_{\mathbb{Q}} \equiv \mathbb{Q}^n \cap \mathcal{D}$. Since $\mathbb{Q}^n$ is countable it follows that $\mathcal{D}_{\mathbb{Q}}$ is countable. Say $\mathcal{D}_{\mathbb{Q}} = \{y^j;\ j \in \mathbb{N},\ y^j \in \mathbb{Q}^n\}$.

We will inductively define the sub-sequence $\{f_{j_k}\}_{k=1}^{\infty}$ so that it converges pointwise on $\mathcal{D}_{\mathbb{Q}}$.

Consider the sequence $\{f_j(y^1)\}_{j=1}^{\infty}$. Since $|f_j| \leq C$ in $\mathcal{D}$ it follows that $|f_j(y^1)| \leq C$. In particular $\{f_j(y^1)\}_{j=1}^{\infty}$ is a bounded sequence of real numbers. We may thus extract a convergent sub-sequence which we will denote $\{f_{1,j}\}_{j=1}^{\infty}$ where the sub-script 1 indicates that the sequence converges at y^1.

Next we make the induction assumption that we have extracted sub-sequences $\{f_{l,j}\}_{j=1}^{\infty}$ for each $l \in \{1,2,3,...,m\}$, such that

1. $\{f_{l,j}\}_{j=1}^{\infty}$ is a sub-sequence of $\{f_{l-1,j}\}_{j=1}^{\infty}$ for $l = 2,3,4,...,m$
2. and $f_{m,j}(y^l)$ converges for $l = 1,2,3,...,m$.

In order to complete the induction we need to show that we can find a sub-sequence $\{f_{m+1,j}\}_{j=1}^{\infty}$ of $\{f_{m,j}\}_{j=1}^{\infty}$ such that $\{f_{m+1,j}(y^{m+1})\}_{j=1}^{\infty}$ converges.

Arguing as before, we see that $\{f_{m,j}(y^{m+1})\}_{j=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$ and we may thus extract a sub-sequence, which we denote $\{f_{m+1,j}\}_{j=1}^{\infty}$, that converges.

By induction it follows that for each $m \in \mathbb{N}$ there exist a sequence $\{f_{m,j}\}_{j=1}^{\infty}$ such that $\{f_{m,j}\}_{j=1}^{\infty}$ is a sub-sequence of $\{f_{m-1,j}\}_{j=1}^{\infty}$ and $\{f_{m,j}(y^m)\}_{j=1}^{\infty}$ is convergent.

Notice that since $\{f_{m,j}\}_{j=1}^{\infty}$ is a sub-sequence of $\{f_{m-1,j}\}_{j=1}^{\infty}$ and $\{f_{m-1,j}(y^l)\}_{j=1}^{\infty}$ converges for $1 \leq l \leq m-1$ it follows that $\{f_{m,j}(y^l)\}_{j=1}^{\infty}$ converges to the same limit for $1 \leq l \leq m-1$. In particular, $\{f_{m,j}(y^l)\}_{j=1}^{\infty}$ converges for all $l \leq m$.

Now we define the sequence $\{f_{j_k}\}_{k=1}^{\infty}$ by a diagonalisation procedure

$$f_{j_k} = f_{k,k}.$$

Noticing that $\{f_{j_k}\}_{k=m}^{\infty} = \{f_{k,k}\}_{k=m}^{\infty}$ is a sub-sequence of $\{f_{m,j}\}_{j=1}^{\infty}$. This follows from the fact that $f_{k,k}$ is an element of the sequence $\{f_{k,j}\}_{j=1}^{\infty}$. But $\{f_{k,j}\}_{j=1}^{\infty}$ is a sub-sequence of $\{f_{m,j}\}_{j=1}^{\infty}$ for $k \geq m$.

We may conclude that $\{f_{j_k}\}_{k=m}^{\infty}$ converges at y^l for all $l \leq k$. But k is arbitrary so $f_{j_k}(y^l)$ converges for every $l \in \mathbb{N}$. This proves step 1.

Step 2: *The sequence $\{f_{j_k}\}_{k=1}^{\infty}$ converges pointwise in $\mathcal{D}$.*

It is enough to show that $\{f_{j_k}(x)\}_{k=1}^{\infty}$ is a Cauchy sequence for every $x \in \mathcal{D}$. To that end we fix an $\epsilon > 0$. We need to show that there exist an $N_\epsilon \in \mathbb{N}$ such that $|f_{j_k}(x) - f_{j_l}(x)| < \epsilon$ for all $k, l > N_\epsilon$.

Since $\{f_{j_k}\}_{k=1}^{\infty}$ is equicontinuous at $x \in \mathcal{D}$ there exist a $\delta_{x,\epsilon/3}$ such that

$$|f_{j_k}(x) - f_{j_k}(y)| < \frac{\epsilon}{3} \quad \text{for all } k \in \mathbb{N}, \tag{76}$$

and $y \in \mathcal{D}$ such that $|x - y| < \delta_{x,\epsilon/3}$.

Moreover since $\mathcal{D}_\mathbb{Q}$ is dense in $\mathcal{D}$ there exist an $y^x \in \mathcal{D}_\mathbb{Q}$ such that $|x - y^x| < \delta_{x,\epsilon/3}$. In step 1 we showed that $f_{j_k}(y)$ was convergent for all $y \in \mathcal{D}_\mathbb{Q}$ in particular it follows that $\{f_{j_k}(y^x)\}_{k=1}^{\infty}$ is a Cauchy sequence. That is, there exist an $N_{y^x,\epsilon/3} \in \mathbb{N}$ such that

$$|f_{j_k}(y^x) - f_{j_l}(y^x)| < \frac{\epsilon}{3} \quad \text{for all } k, l > N_{y^x,\epsilon/3}. \tag{77}$$

From (76) and (77) we can deduce that

$$|f_{j_k}(x) - f_{j_l}(x)| \leq |f_{j_k}(x) - f_{j_k}(y^x)| + |f_{j_l}(x) - f_{j_l}(y^x)| + |f_{j_k}(y^x) - f_{j_l}(y^x)| < \epsilon,$$

for all $k, l > N_{y^x,\epsilon/3}$. It follows that $\{f_{j_k}(x)\}_{k=1}^{\infty}$ is a Cauchy sequence and this finishes the proof of step 2.

Step 3: *Define $f_0(x) = \lim_{k\to\infty} f_{j_k}(x)$, then $f_0 \in C(\mathcal{D})$.*

Since $f_{j_k}(x)$ is convergent for every $x \in \mathcal{D}$ by step 2 it follows that f_0 is well defined in $\mathcal{D}$. To show continuity we need to show that for every $x \in \mathcal{D}$ and $\epsilon > 0$ there exist a $\delta_\epsilon > 0$ such that

$$|f_0(x) - f_0(y)| < \epsilon$$

for every $y \in \mathcal{D}$ such that $|x - y| < \delta_\epsilon$. By equicontinuity there exist a $\delta_{x,\epsilon/3}$ such that

$$|f_{j_k}(x) - f_{j_k}(y)| < \frac{\epsilon}{3} \tag{78}$$

for every $y \in \mathcal{D}$ such that $|x - y| < \delta_{x,\epsilon/3}$ and all $j \in \mathbb{N}$.

Also by step 2 there exist an $N_{x,\epsilon/3}$ such that

$$|f_0(x) - f_{j_k}(x)| < \frac{\epsilon}{3} \tag{79}$$

for all $k \geq N_{x,\epsilon/3}$. And an $N_{y,\epsilon/3}$ such that

$$|f_0(y) - f_{j_k}(y)| < \frac{\epsilon}{3} \tag{80}$$

for all $k \geq N_{y,\epsilon/3}$.

From (78), (79) and (80) we can deduce that for $y \in \mathcal{D}$ such that $|x - y| < \delta_{x,\epsilon/3}$

$$|f_0(x) - f_0(y)| \leq |f_0(x) - f_{j_k}(x)| + |f_0(y) - f_{j_k}(y)| + |f_{j_k}(x) - f_{j_k}(y)| < \epsilon$$

if $k > \max(N_{x,\epsilon/3}, N_{y,\epsilon/3})$.

This proves step 3.

Step 4: $\{f_{j_k}\}_{k=1}^{\infty}$ *converges uniformly on compact sets.*

We fix a compact set $K \subset \mathcal{D}$. We need to show that for every $\epsilon > 0$ there exist an N_ϵ such that when $k > N_\epsilon$ then $|f_0(x) - f_{j_k}(x)| < \epsilon$ for all $x \in K$.

Notice that by equicontinuity there exist a $\delta_{x,\epsilon/3}$ for each $x \in K$ such that for all $k \in \mathbb{N}$

$$|f_{j_k}(x) - f_{j_k}(y)| < \frac{\epsilon}{3} \tag{81}$$

for all $y \in B_{\delta_{x,\epsilon/3}}(x) \cap \mathcal{D}$.

Notice that the balls $B_{\delta_{x,\epsilon/3}}(x)$ forms an open cover of K: $K \subset \cup_{x\in K} B_{\delta_{x,\epsilon/3}}(x)$. Since K is compact there exist a finite sub-cover $B_{\delta_{x^l,\epsilon/3}}(x^l)$, for $l = 1, 2, 3, ..., l_0$ for some $l_0 \in \mathbb{N}$. That is $K \subset \cup_{l=1}^{l_0} B_{\delta_{x^l,\epsilon/3}}(x^l)$.

Also, using that $\lim_{k\to\infty} f_{j_k}(x^l) = f_0(x^l)$, we see that there exist an $N_{x^l,\epsilon/3}$ such that

$$|f_{j_i}(x^l) - f_{j_k}(x^l)| < \frac{\epsilon}{3} \tag{82}$$

for all $i, k > N_{x^l,\epsilon/3}$. We choose $N_\epsilon = \max\left(N_{x^1,\epsilon/3}, N_{x^2,\epsilon/3}, ..., N_{x^{l_0},\epsilon/3}\right)$.

Since $K \subset \cup_{l=1}^{l_0} B_{\delta_{x^l,\epsilon/3}}(x^l)$ it follows that for every $x \in K$ that $x \in B_{\delta_{x^l,\epsilon/3}}(x^l)$ for some l. Using this and (81) and (82) we see that

$$|f_{j_i}(x) - f_{j_k}(x)| \le |f_{j_i}(x) - f_{j_i}(x^l)| + |f_{j_k}(x) - f_{j_k}(x^l)| + |f_{j_i}(x^l) - f_{j_k}(x^l)| < \tag{83}$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

for all $k \ge N_\epsilon$. Taking the limit $i \to \infty$ in (83) we see that

$$|f_0(x) - f_{j_k}(x)| < \epsilon$$

for all $k > N_\epsilon$. This finishes the proof of the Theorem. □

B Bibliographic note.

All the theory in these notes are classical and may be found in many books on partial differential equations. Our development is very similar to the one in Folland [2]. But some proofs and calculations comes from [1]. Some of the functional analytic theory comes from [3]. A more advanced exposition of the theory in these notes can be found in [4].

References

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