Some further exercises and Problems to work on Integration Theory, Fall, 2020

this will be regularly updated. You should prioritize working on the exercises in the course notes written by me as well as the exercises listed in Folland. The list here should be viewed as a supplementary list of hopefully interesting, perhaps more challenging, exercises.

September 18, 2020

1. Consider a σ-finite measure space $(X, \mathcal{M}, \mu)$ and let $(A_\alpha)_{\alpha \in I}$ be disjoint subsets of X. Show that only countably many of the A_α's can have positive μ-measure. (You will need to look ahead in the text for the definition of a σ-finite measure.)

2. Prove or disprove: If both A and B are not Lebesgue measurable, then $A \cup B$ is not Lebesgue measurable.

3. Compare the following two statements for $f : [0,1] \to R$. Which implications hold?

1. f is continuous a.e.

2. There exists $g : [0,1] \to R$ which is continuous so that $f = g$ a.e.

4. Find a nonmeasurable set in $[0,1]$ whose outer measure is at most .0001. (This should only require a very minor variation on the nonmeasurable set constructed in the notes.)

5. For $p \geq 1$, let $L^p(X, \mathcal{B}, \mu) := \{f \text{ measurable} : \int_X |f|^p d\mu < \infty\}$. If $p > q$, what is the relationship between $L^p(X, \mathcal{B}, \mu)$ and $L^q(X, \mathcal{B}, \mu)$ in the following three cases.

1. Lebesgue measure on $[0,1]$.

2. Lebesgue measure on R.

3. Counting measure on N.

6. Let X be a metric space and μ be a finite measure on the Borel sets. (Borel sets are the smallest σ-algebra containing the open sets). Show that for any Borel set E,

$$\mu(E) = \inf\{\mu(O) : O \supseteq E \text{ and O open }\} = \sup\{\mu(C) : C \subseteq E \text{ and C closed }\}.$$

(Hint: Let $\mathcal{E}$ be the collection of Borel sets E where the theorem is true and show that $\mathcal{E}$ is a sigma-algebra and contains the set of closed sets.)

7. Let f_n be the function on $[0, \infty)$ defined by

$$f_n(x) = \frac{x}{1 + x^n}.$$

a. What is the pointwise limit of the sequence (f_n)?

b. Is the convergence occuring uniformly on $[0, \infty)$?

c. Determine

$$\lim_{n \to \infty} \int_0^\infty f_n(t) dt.$$

Hint: Do not try to compute these integrals but rather use some convergence theorem.

8. Let μ_n be a sequence of measures on $(X, \mathcal{M})$ which are increasing in that for all $A \in \mathcal{M}$, $\mu_n(A)$ is an increasing sequence in n. Show that

$$\mu(A) := \lim_{n \to \infty} \mu_n(A)$$

is a measure on $(X, \mathcal{M})$.

9. Construct a nonnegative function on $[0, 1]$ which is finite everywhere and where $\int_a^b f(x) = \infty$ for all $0 \leq a < b \leq 1$. Hint: Borel-Cantelli Lemma. (One does not need to know the notion of a Lebesgue integral for this; the standard notion of integration from calculus suffices.)

10. Let m be Lebesgue measure on R and let μ be a sigma-finite measure on $(R, \mathcal{B})$. Let A be a Borel set with $m(A) = 0$ but $\mu(A) > 0$.

1. Show that m-a.e. translate of A has 0 measure with respect to μ; i.e.

$$m(\{x \in R : \mu(A + x) > 0\}) = 0.$$

(Hint: Fubini's Theorem applied to a well chosen product space and a well chosen set.)

2. Show that m-a.e. scaling of A has 0 measure with respect to μ; i.e.

$$m(\{x \in (0, \infty) : \mu(xA) > 0\}) = 0.$$

(Hint: Same as above).

11. Find an example of functions f and g which are defined on $[0, 1]$ such that (1) the image of g is contained in $[0, 1]$ (which allows us to define $f \circ g$), (2) both f and g have finite variation but (3) $f \circ g$ does not have finite variation.

(Hint: Use the fact that for a function on $[0, 1]$ which is continuously differentiable on $[\epsilon, 1]$ for each $\epsilon > 0$, one has that f has finite variation if and only if

$$\int_0^1 |f'(x)|dx < \infty(\text{ also equivalently to being absolutely continuous }))$$

12. Assume that $f : [0, 1] \to R$ has a finite derivative at every point (but you cannot assume the derivative is continuous). Show that there is a nontrivial closed interval $I \subseteq [0, 1]$ so that f is absolutely continuous on I. (Hint: You can use the (nontrivial and interesting) fact that a function which is a pointwise limit of continuous functions contains points of continuity.)

13. The following is a converse to Lebesgue's differentiation theorem in 1-d. Show that if $A \subseteq [0, 1]$ has Lebesgue measure 0, then there exists a continuous monotone function f on $[0, 1]$ whose derivative is infinite at every point of A.

Hints.

Step 1: Show that the Borel-Cantelli is essentially sharp in the sense that given $A \subseteq [0, 1]$ with Lebesgue measure 0, there exist a collection E_n of open intervals such that

$$\sum m(E_n) < \infty$$

but

$$A \subseteq \limsup E_n.$$

Step 2: Let F_n be the distribution function for the measure which is Lebesgue measure on E_n. (Note $F_n(0) = 0$ and $F_n(1) = m(E_n)$.) Let now $f = \sum_n F_n$.